Auctions & Mechanism Design Basics

Joseph Chuang-Chieh Lin

Dept. CSIE, Tamkang University, Taiwan

- We study about a kind of science of *rule-making*.
- To make it simple, we first consider single-item auctions.
- We will go over some basics about first-price auctions and second-price auctions.
- Also, we will talk about
  - incentive guarantees,
  - strong performance guarantees, and
  - computational efficiency

  in an auction.
- We will end the discussion with Myerson's Lemma.

Outline

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  - Her maximum willingness-to-pay for it.
  - v_i is private.
    - Unknown to the seller and other bidders.

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  - If she wins $\Rightarrow$ $v_i - p$.

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Sealed-Bid Auctions

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(i) Each bidder i privately communicates a bid b_i to the seller—in a sealed envelope.

(ii) The seller decides who gets the item (if any).

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- Step (ii): The selection rule. We consider giving the item to the **highest** bidder.

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- For a bidder: Hard to figure how to bid.
- For the seller: Hard to predict what will happen.

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  - Would your answer change if you knew there were two other bidders rather than one?

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eBay/Yahoo auction

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    whichever comes first.
  - For example, if the highest other bid is \$90.
    You only pay \$90 $+ \epsilon$ for some small increment ϵ.

  $\approx$ highest other bid!

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Second-Price/Vickrey Auction

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- Is such a strategy a dominant strategy?
  - The strategy is guaranteed to **maximize** a bidder's utility **no matter what other bidders do**.

Truthfully Bidding Is Dominant Here

Proposition (Incentives in Second-Price Auctions)

In a second-price auction, every bidder i has a dominant strategy: set the bid $b_i = v_i$, equal to her private valuation.

Proof of the Proposition

- Fix a bidder i with valuation v_i.
- $\boldsymbol{b}$: the vector of all bids.
- $\boldsymbol{b}_{-i}$: the vector of $\boldsymbol{b}$ with b_i removed.

* **Goal**: Show that bidder i's utility is maximized by setting $b_i = v_i$.

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  - $\because$ bidder i wins and bids her true valuation v_i, so $p \leq v_i \Rightarrow v_i - p \geq 0$.

Second-Price Single-Item Auctions are "ideal"

Definition (Dominant-Strategy Incentive Compatible)

An auction is dominant-strategy incentive compatible (DSIC) if

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Social Welfare

The social welfare of an outcome of a single-item auction is

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where $\sum_{i=1}^{n} x_i \leq 1$; $x_i = 1$ if bidder i wins and 0 if she loses.

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- So such an auction is welfare maximizing if bids are truthful.

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Theorem

A second-price single-item auction satisfies:

(1) DSIC.

(2) Welfare maximizing.

(3) It can be implemented in polynomial time.

In fact, (3) is linear.

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Theorem

A second-price single-item auction satisfies:

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(2) Welfare maximizing. (strong performance guarantees)

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Background

The Social Dilemma (2020) - Trailer

- Web search results:
  - relevant to your query (by an algorithm, e.g., PageRank).
  - pops out a list of sponsored links.
    - They are paid by advertisers.
- Every time you give a search query into a search engine, an auction is run in real time to decide
  - which advertiser's links are shown,
  - how these links are arranged visually,
  - what the advertisers are charged.

Multiple Items for Sponsored Search Auctions

- Let's say the items for sale are k "slots" on a search results page.
- Bidders: the advertisers who have a bid on the keyword that was searched on.
  - On the keyword, "university", NTU, NYCU, NCKU, TKU, etc., might be the bidders.

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  - On the keyword, "camera", Nikon, Canon, Sony, etc., might be the bidders.
  - On the keyword, "SUV", Toyota, Ford, Honda, Porsche, etc., might be the bidders.
- Let's say the items are not identical.
  - Higher slots are more valuable. What do you think?

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- Consider the click-through-rates (CTRs) α_j of slot j.
  - The probability that the user clicks on this slot.
  - Assumption: $\alpha_1 \geq \alpha_2 \geq \dots \alpha_k$.

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  - The CTR of advertiser i in slot j: $\beta_i \alpha_j$.
- The expected value derived by advertiser i from slot j: $v_i \alpha_j$
- The social welfare is $\sum_{i=1}^{n} v_i x_i$.
  - x_i: the CTR of the slot to which bidder i is assigned.
    - $x_i = 0$: bidder i is not assigned to a slot.
  - Each slot can only be assigned to one bidder; each bidder gets only one slot.

Our Design Approach

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  - If the payments are not just right, then the strategic bidders will game the system.

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Design Steps

(a): Assume that the bidders bid truthfully. Then, how should we assign bidders to slots so that property (2) and (3) holds?

(b): Given the answer of above, how should we set selling prices so that property (1) holds?

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- You can prove that this assignment achieves the maximum social welfare as an exercise.

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- There is an analog of the second-price rule.
  - DSIC.
  - ⋆ Myerson's lemma.
    - A powerful and general tool for implementing this second step.

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Single-Parameter Environments

Consider a more generalized and abstract setting:

Single-Parameter Environments

- n agents (e.g., bidders).
- A private valuation $v_i \geq 0$ for each agent i (per unit of stuff).
- A feasible set $X = \{(x_1, x_2, \ldots, x_n) \mid x_i \in \mathbb{R}\} \subseteq \mathbb{R}^n$.
  - x_i: amount of stuff given to agent i.

Single-Parameter Environments (Examples)

- Single-item auction:
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- k-Unit auction:
  - k identical items, $\sum_{i=1}^{n} X_i \leq k$, and $x_i \in \{0, 1\}$ for each i.
- Sponsored Search Auction:
  - X: the set of n-vectors $\Leftrightarrow$ assignments of bidders to slots.
  - Each slot (resp., bidder) is assigned to ≤ 1 bidder (resp., slot).
  - The component $x_i = \alpha_j$: bidder i is assigned to slot j.
    - α_j: the click-through rate of slot j.
    - Assume that the quality score $\beta_i = 1$ for all i.

Allocation and Payment Rules

Choices to make in a sealed-bid auction

- Collect bids $\boldsymbol{b} = (b_1, \ldots, b_n)$.
- Allocation Rule: Choose a feasible $\boldsymbol{x}(\boldsymbol{b}) \in X \subseteq \mathbb{R}^n$.
- Payment Rule: Choose payments $\boldsymbol{p}(\boldsymbol{b}) \in \mathbb{R}^n$.

- A *direct-revelation mechanism*.

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- A *direct-revelation mechanism*.
- Example of *indirect mechanism*: iterative ascending auction.

Allocation and Payment Rules (contd.)

With allocation rule $\boldsymbol{x}$ and payment rule $\boldsymbol{p}$,

- agent i receives utility $u_i(\boldsymbol{b}) = v_i \cdot x_i(\boldsymbol{b}) - p_i(\boldsymbol{b})$.
- $p_i(\boldsymbol{b}) \in [0, b_i \cdot x_i(\boldsymbol{b})]$.
  - $p_i(\boldsymbol{b}) \geq 0$: prohibiting the seller from paying the agents.
  - $p_i(\boldsymbol{b}) \leq b_i \cdot x_i(\boldsymbol{b})$: a truthful agent receives nonnegative utility.

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The Myerson's Lemma

Definition (Implementable Allocation Rule)

An allocation rule $\boldsymbol{x}$ for a single-parameter environment is implementable if there is a payment rule $\boldsymbol{p}$ such that the direct-revelation mechanism $(\boldsymbol{x}, \boldsymbol{p})$ is DSIC.

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So, how about awarding the item to the second-highest bidder?

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The rules that extend to DSIC mechanisms.

Definition (Monotone Allocation Rule)

An allocation rule $\boldsymbol{x}$ for a single-parameter environment is monotone if for every agent i and bids $\boldsymbol{b}_{-i}$ by other agents, the allocation $x_i(z, \boldsymbol{b}_{-i})$ to i is nondecreasing in her bid z.

Bidding higher can only get you more stuff!
So, how about awarding the item to the second-highest bidder?
You raise your bid, you might lose the chance of getting it!

Outline

The Myerson's Lemma

Theorem (Myerson's Lemma)

Fix a single-parameter environment.

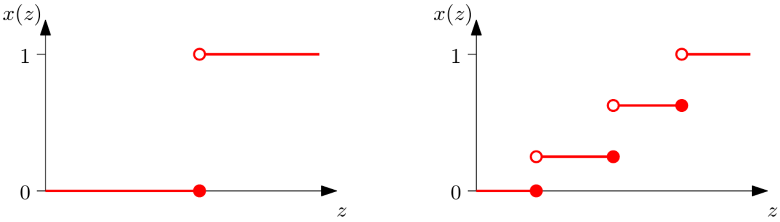
(i) An allocation rule $\boldsymbol{x}$ is implementable if and only if it is monotone.

(ii) If $\boldsymbol{x}$ is monotone, then there is a unique payment rule for which the direct-revelation mechanism $(\boldsymbol{x}, \boldsymbol{p})$ is DSIC and $p_i(\boldsymbol{b}) = 0$ whenever $b_i = 0$.

(iii) The payment rule in (ii) is given by an explicit formula.

"Monotone" is more operational.

Allocation curves: allocation as a function of bids



(a) 0-1 Monotone Curve (b) Piecewise Constant Monotone Curve

Figures from Tim Roughgarden's lecture notes.

Constraints from DSIC

Consider $0 \leq z < y$.

Say agent i has a private valuation z and free to submit a false bid y or agent i has a private valuation y and free to submit a false bid z

DSIC: Bidding truthfully brings maximum utility.

$$\begin{aligned} z \cdot x(z) - p(z) &\geq z \cdot x(y) - p(y) \\ y \cdot x(y) - p(y) &\geq y \cdot x(z) - p(z) \end{aligned}$$

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$\Rightarrow$ every implementable allocation rule is monotone (why?)

Case: x is a piecewise constant function

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- Try: fix z and let y tend to z.

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  $\Rightarrow$ left-hand and right-hand sides $\to 0$ if there is no jump in x at z.

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$$p_i(b_i, \boldsymbol{b}_{-i}) = \sum_{j=1}^{\ell} z_j \cdot [\text{ jump in } x_i(\cdot, \boldsymbol{b}_{-i}) \text{ at } z_j],$$

where $z_1, \ldots, z_\ell$ are breakpoints of $x_i(\cdot, \boldsymbol{b}_{-i})$ in the range $[0, b_i]$.

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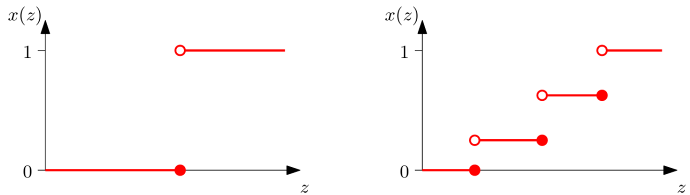

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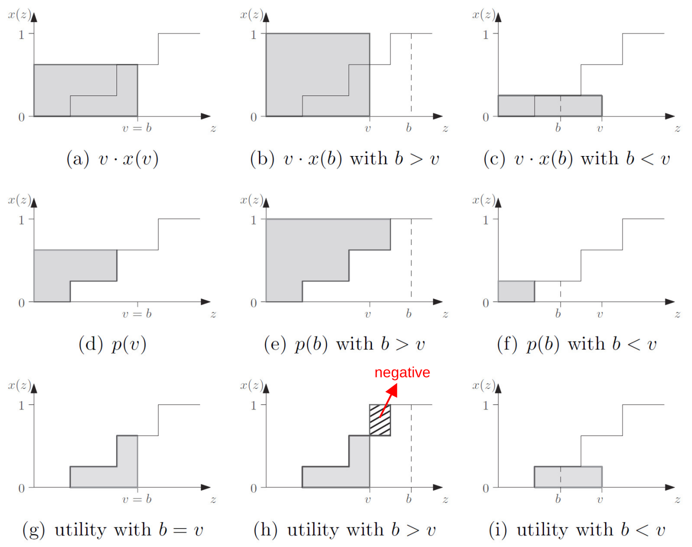

(a) $v \cdot x(v)$ (b) $v \cdot x(b)$ with $b > v$ (c) $v \cdot x(b)$ with $b < v$

(d) $p(v)$ (e) $p(b)$ with $b > v$ (f) $p(b)$ with $b < v$

(g) utility with $b = v$ (h) utility with $b > v$ (i) utility with $b < v$

Case: x is a monotone function

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- Suppose x is differentiable.
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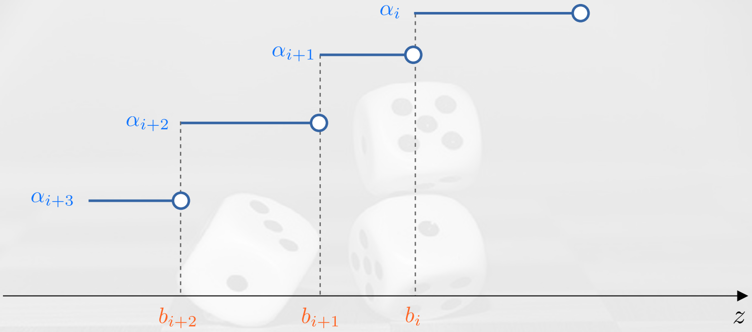
$$p_i(b_i, \boldsymbol{b}_{-i}) = \int_0^{b_i} z \cdot \frac{d}{dz} x_i(z, \boldsymbol{b}_{-i}) dz.$$

Outline

Apply to Sponsored Search Auction

The allocation rule is piecewise.

re-index the bidders: $b_1 \geq b_2 \geq \ldots \geq b_n$.



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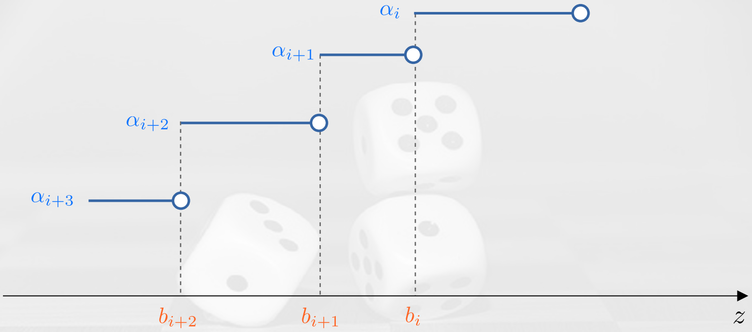

$$p_i(\boldsymbol{b}) = \sum_{j=i}^{k} b_{j+1}(\alpha_j - \alpha_{j+1}).$$

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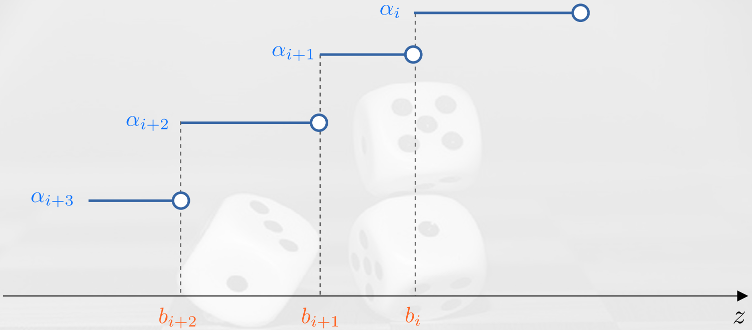

$$p_i(\boldsymbol{b}) = \sum_{j=i}^{k} b_{j+1} \frac{\alpha_j - \alpha_{j+1}}{\alpha_i}$$ (scaled per click).

Exercise 1 (5%)

- Recall that in the model of sponsored search auctions:
  - There are k slots, the jth slot has a click-through rate (CTR) of α_j (nonincreasing in j).
  - The utility of bidder i in slot j is $\alpha_j(v_i - p_j)$, where v_i is the private value-per-click of the bidder and p_j is the price charged per-click in slot j.
- The Generalized Second Price (GSP) Auction is defined as follows:

Exercise 1 (5%) (contd.)

The Generalized Second Price (GSP) Auction

1. Rank advertisers from highest to lowest bid; assume without loss of generality that $b_1 \geq b_2 \geq \cdots \geq b_n$.
2. For $i = 1, 2, \ldots, k$, assign the ith bidder to the i slot.
3. For $i = 1, 2, \ldots, k$, charge the ith bidder a price of b_{i+1} per click.

(a) Prove that for every $k \geq 2$ and sequence $\alpha_1 \geq \cdots \geq \alpha_k > 0$ of CTRs, the GSP auction is NOT DSIC. (*Hint: Find out an example.*)

(b) A bid profile $\boldsymbol{b}$ with $b_1 \geq \cdots \geq b_n$ is envy-free if for every bidder i and slot $j \neq i$,

$$\alpha_i(v_i - b_{i+1}) \geq \alpha_j(v_i - b_{j+1}).$$

Please verify that every envy-free bid profile is an equilibrium.