Equilibrium Concepts

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Lecture Notes in Algorithmic Game Theory

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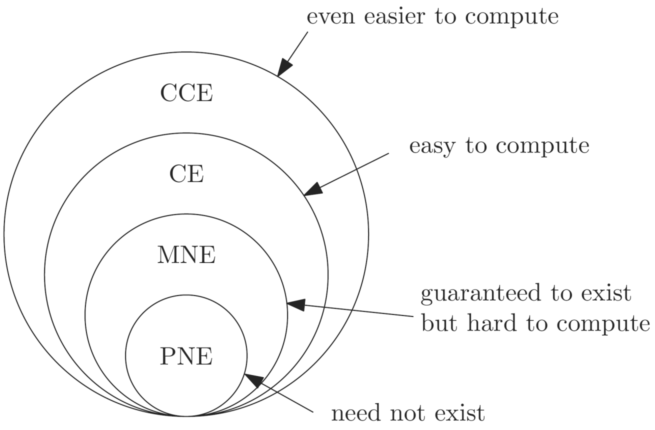
Outline

A hierarchy of equilibrium concepts



Outline

Cost-Minimization Games

A cost-minimization game has the following ingredients:

- a finite number of k agents;
- a finite set S_i of pure strategies for each agent i;
- a nonnegative cost function $C_i(\mathbf{s})$ for each agent i.
  - $\mathbf{s} \in S_1 \times S_2 \times \cdots \times S_k$: a strategy profile or outcome.

For example, the network creation game.

Payoff-Maximization Games

A **payoff-maximization** game has the following ingredients:

- a finite number of k agents;
- a finite set S_i of pure strategies for each agent i;
- a nonnegative payoff function $\pi_i(\mathbf{s})$ for each agent i.
  - $\mathbf{s} \in S_1 \times S_2 \times \cdots \times S_k$: a strategy profile or outcome.

For example, the Rock-Paper-Scissors game, two-party election game, etc.

Outline

Pure Nash Equilibrium (PNE)

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A strategy profile **s** of a cost-minimization game is a pure Nash equilibrium (PNE) if for every agent $i \in \{1, 2, \ldots, k\}$ and every unilateral deviation $s_i' \in S_i$,

$$C_i(\mathbf{s}) \leq C_i(s_i', \mathbf{s}_{-i}).$$

- $\mathbf{s}_{-i}$: the vector **s** with the ith component removed.

Outline

Mixed Nash Equilibrium (MNE)

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Distributions $\sigma_1, \ldots, \sigma_k$, over strategy sets $S_1, \ldots, S_k$ respectively, of a cost-minimization game constitute a mixed Nash equilibrium (MNE) if for every agent $i \in \{1, 2, \ldots, k\}$ and every unilateral deviation $s_i' \in S_i$,

$$\mathbf{E}_{\mathbf{s}\sim\sigma}[C_i(\mathbf{s})] \leq \mathbf{E}_{\mathbf{s}\sim\sigma}[C_i(s_i', \mathbf{s}_{-i})].$$

- σ: the product distribution $\sigma_1 \times \cdots \times \sigma_k$.

Product of Mixed Strategies

Player 2

q_1 q_2 q_3 → probabilities

Player 1

		rock	scissors	paper
p_1	rock	$p_1 q_1$ 0, 0	$p_1 q_2$ 1, −1	$p_1 q_3$ −1, 1
p_2	scissors	$p_2 q_1$ −1, 1	$p_2 q_2$ 0, 0	$p_2 q_3$ 1, −1
p_3	paper	$p_3 q_1$ 1, −1	$p_3 q_2$ −1, 1	$p_3 q_3$ 0, 0

↓ probabilities

$$\begin{cases} p_1 + p_2 + p_3 = 1. \\ q_1 + q_2 + q_3 = 1. \end{cases}$$

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Correlated Equilibrium (CE)

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A distribution σ on the set $S_1 \times \ldots \times S_k$ of outcomes of a cost-minimization game is a correlated equilibrium (CE) if for every agent $i \in \{1, 2, \ldots, k\}$ and every unilateral deviation $s'_i \in S_i$,

$$\mathbf{E}_{\mathbf{s}\sim\sigma}[C_i(\mathbf{s}) \mid s_i] \leq \mathbf{E}_{\mathbf{s}\sim\sigma}[C_i(s'_i, \mathbf{s}_{-i}) \mid s_i].$$

Stop or Go?

Matrix of costs

	Stop	Go
Stop	1, 1	1, 0
Go	0, 1	5, 5

- Two PNEs.

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	Stop	Go
Stop	prob. $= 0$ 1, 1	prob. $= 1/2$ 1, 0
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- A CE for example.
- Cannot correspond to a MNE.

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Game of Chicken

- A.k.a. Hawk-Dove Game.
  - A model of conflict for two players.

	Dare	Chicken
Dare	0, 0	7, 2
Chicken	2, 7	6, 6

- Two PNE & One MNE.
- The expected utility of each player in the MNE:

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Game of Chicken

- A correlated equilibrium.
  - Check that it is an equilibrium if a player is assigned “Dare”.
  - Check that it is an equilibrium if a player is assigned “Chicken Out”.

	Dare	Chicken
Dare	prob. $= 0$ 0, 0	prob. $= 1/3$ 7, 2
Chicken	prob. $= 1/3$ 2, 7	prob. $= 1/3$ 6, 6

- The expected utility for each player: $7 \cdot (1/3) + 2 \cdot (1/3) + 6 \cdot (1/3) = 5.$

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CE $\subseteq$ CCE?

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\mathbf{E}_{\mathbf{s}\sim\sigma}[C_i(\mathbf{s})] &= \sum_{a\in S_i} \mathbf{E}_{\mathbf{s}\sim\sigma}[C_i(\mathbf{s}) \mid s_i = a] \Pr[s_i = a] \\
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\end{aligned}$$

CCE Example

	A	B	C
A	prob. $= 1/3$ 1, 1	$-1, -1$	0, 0
B	$-1, -1$	prob. $= 1/3$ 1, 1	0, 0
C	0, 0	0, 0	prob. $= 1/3$ $-1.1, -1.1$

- The payoff for each player (playing according to this distribution): $\frac{1}{3} \cdot 1 + \frac{1}{3} \cdot 1 - \frac{1}{3} \cdot 1.1 = 0.3$.
- A player playing fixed A or B while the opponent randomized according to this distribution: $\frac{1}{3} \cdot 1 - \frac{1}{3} \cdot 1 + \frac{1}{3} \cdot 0 = 0$.
- A player playing fixed C while the opponent randomized according to this distribution: $\frac{1}{3} \cdot 0 + \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot (-1.1) < 0$.

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  $\Rightarrow$ deviation is possible.
  - Not a CE.

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