Network Creation Games

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Lecture Notes in Algorithmic Game Theory

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Outline

1. Introduction
2. Classical Network Creation Game
   - Computational Hardness
   - Price of Anarchy Bounds for the Classic Model
   - Some $O(1)$ upper bounds for $\alpha = O(\sqrt{n})$
3. Social Network Creation Game (SNCG)
   - Properties
   - The PoA of SNCG
   - Experimental Results
4. Reference

Outline

Network creation games

- First introduced in *PODC*'03.

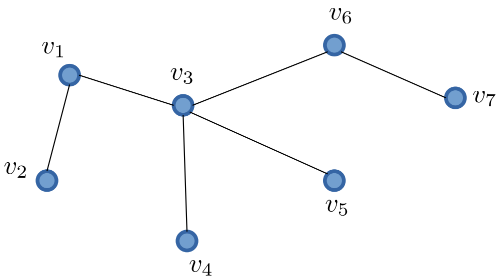
Alex Fabrikant

Ankur Luthra

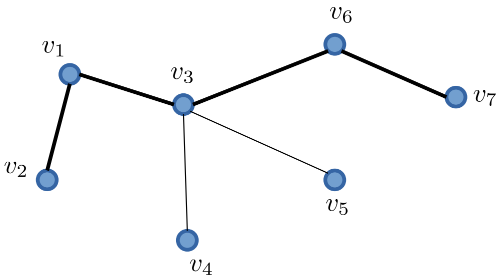
Elitza Maneva

Christos H. Papadimitriou

Scott Shenker


v_1
v_2
v_3
v_4
v_5
v_6
v_7

diameter: length of the longest short-test path $\Rightarrow$ 4.

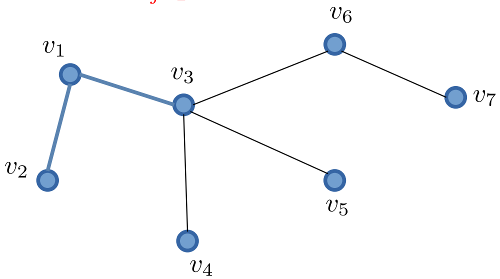


edge cost : $\alpha := 1$

distance cost of v_1 : $d_G(v_1, v_2) + d_G(v_1, v_3) + d_G(v_1, v_4) + d_G(v_1, v_5) + d_G(v_1, v_6) + d_G(v_1, v_7) = 11.$

edge cost of v_1 : $2\alpha = 2$

Total cost of $v_1 = \alpha \cdot |N_{v_1}| + \sum_{j=2}^{7} d_G(1, j) = 13.$



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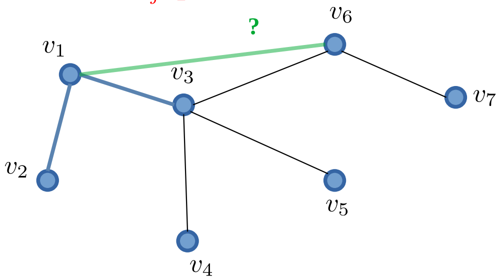

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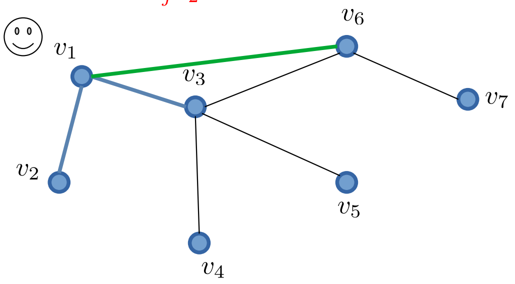
edge cost of v_1 : $3\alpha = 3.$

Total cost of $v_1 = \alpha \cdot |N_{v_1}| + \sum_{j=2}^{7} d_G(1, j) = 12.$

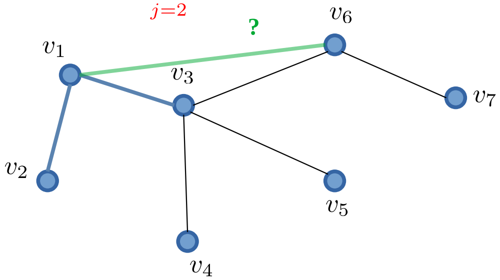


edge cost : $\alpha := 2 \Rightarrow$ increased!

distance cost of v_1 : $d_G(v_1, v_2) + d_G(v_1, v_3) + d_G(v_1, v_4) + d_G(v_1, v_5) + d_G(v_1, v_6) + d_G(v_1, v_7) = 11.$

edge cost of v_1 : $2\alpha = 4$, $3\alpha = 6$.

Total cost of $v_1 = \alpha \cdot |N_{v_1}| + \sum_{j=2}^{7} d_G(1, j) = 15$. Or 15 if v_1 connects to v_6



Network creation games [Fabrikant *et al.* @*PODC*'03]

- n players: $1, 2, \ldots, n$.
- s_i: specified by a subset of $\{1, 2, \ldots, n\} \setminus \{i\} = [n] \setminus \{i\}$ as the strategy of player i.
  - The set of neighbors where player i forms a link (edge).
  - edge cost: α.
- $G_{\mathbf{s}}$: the undirected graph with vertex set $[n]$ and edges corresponding to $\mathbf{s} = \langle s_1, s_2, \ldots, s_n \rangle$.
  - $G_{\mathbf{s}}$: an equilibrium graph (when the context is clear).
- $G_{\mathbf{s}}$ has an edge $\{i, j\}$ if either $i \in s_j$ or $j \in s_i$.
- $d_{G_{\mathbf{s}}}(i, j)$: the distance between i and j in $G_{\mathbf{s}}$.

Outline

Network creation games: the classical model

The Classical Model [Fabrikant *et al.* @*PODC*'03]

$$c_i(s) = \alpha|s_i| + \sum_{j=1}^{n} d_{G[\mathbf{s}]}(i,j).$$

- The (total) social cost is $c(s) = \sum_{i=1}^{n} c_i(s)$.

NP-Hardness

Fabrikent *et al.* @*PODC*'03

Given $\mathbf{s} \in S_0 \times \cdots S_{n-1}$ and $i \in [n]$, it is NP-hard to compute the best response of i.

- Reduction from the DOMINATING SET problem.

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- Reduction from the DOMINATING SET problem.

Dominating set

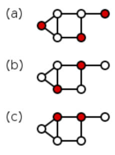
From Wikipedia, the free encyclopedia

For Dominator in control flow graphs, see Dominator (graph theory).

In graph theory, a **dominating set** for a graph $G = (V, E)$ is a subset D of V such that every vertex not in D is adjacent to at least one member of D. The **domination number** $\gamma(G)$ is the number of vertices in a smallest dominating set for G.

The **dominating set problem** concerns testing whether $\gamma(G) \le K$ for a given graph G and input K; it is a classical NP-complete decision problem in computational complexity theory.[1] Therefore it is believed that there may be no efficient algorithm that finds a smallest dominating set for all graphs, although there are efficient approximation algorithms, as well as both efficient and exact algorithms for certain graph classes.

Figures (a)–(c) on the right show three examples of dominating sets for a graph. In each example, each white vertex is adjacent to at least one red vertex, and it is said that the white vertex is *dominated* by the red vertex. The domination number of this graph is 2: the examples (b) and (c) show that there is a dominating set with 2 vertices, and it can be checked that there is no dominating set with only 1 vertex for this graph.



Dominating sets (red vertices).

Outline

The General PoA Upper Bound

Theorem [Fabrikant et al.@*PODC*'03]

The PoA for the sum network creation game is $O(\sqrt{\alpha})$ for all α.

Price of Anarchy for $\alpha < 1$

E: the set of edges in $G_{\mathbf{s}}$.

- **Recall:** The social cost is

$$c(s) = \sum_{i=1}^{n} c_i(s) = \alpha|E| + \sum_{i,j} d_G(i,j)$$
$$\geq \alpha|E| + 2|E| + 2(n(n-1) - 2|E|)$$
$$= 2n(n-1) + (\alpha - 2)|E|.$$

- $\alpha < 1$:
  - the social optimum: the complete graph.
  - ⋆ It's also a NE ($\therefore$ PoA $= 1$).

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Price of Anarchy for $1 \leq \alpha < 2$ (1/2)

- $1 \leq \alpha < 2$:
  - The social optimum: still the complete graph (i.e., K_n).
  - Any NE must be connected and has diameter ≤ 2.
  - ⋆ K_n is NOT the ONLY NE.
  - ⋆ The worst NE: a star.
    - $\alpha \cdot |E| + |E| \cdot 2 \cdot 1 + \left(\binom{n}{2} - |E|\right) \cdot 2 \cdot 2 = (\alpha - 2) \cdot |E| + 2n(n-1)$.

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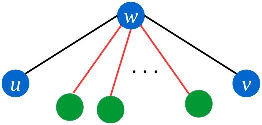


Price of Anarchy for $1 \le \alpha < 2$ (2/2)

- $1 \le \alpha < 2$:
  - The social optimum: still the complete graph (i.e., K_n).
  - Any NE must be connected and has diameter ≤ 2.
  - ⋆ K_n is NOT a NE.
  - ⋆ The worst NE: a star.
    - $\alpha \cdot |E| + |E| \cdot 2 \cdot 1 + \left(\binom{n}{2} - |E|\right) \cdot 2 \cdot 2 = (\alpha - 2) \cdot |E| + 2n(n-1)$.
  - $$\begin{aligned} \text{PoA} &= \frac{C(\text{star})}{C(K_n)} = \frac{(\alpha-2)\cdot(n-1) + 2n(n-1)}{\alpha\binom{n}{2} + 2\cdot\binom{n}{2}\cdot 1} \\ &= \frac{4}{2+\alpha} - \frac{4-2\alpha}{n(2+\alpha)} \\ &< \frac{4}{3}. \end{aligned}$$

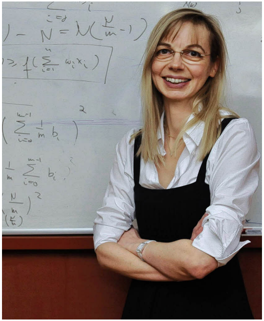
Useful Lemma by Albers *et al.*@SODA 2006

Lemma 1 [Albers et al. @*SODA*'06]

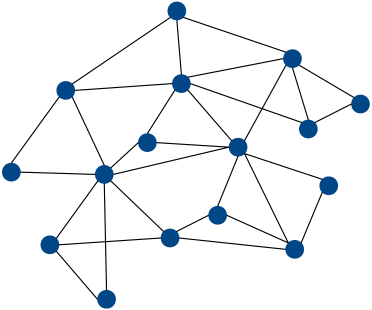
For any Nash equilibrium s and any vertex v_0 in G_s,

$$c(s) \leq 2\alpha(n-1) + n \cdot \mathrm{Dist}(v_0) + (n-1)^2.$$

- $\mathrm{Dist}(v_0) = \sum_{v \in V(G_s)} d_s(v_0, v)$.

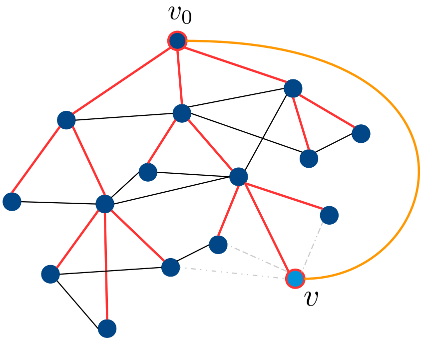


Sketch of proving Lemma 1



- A graph G_s corresponding to a NE s.

Sketch of proving Lemma 1



- $T(v_0)$: the shortest-path tree rooted at v_0.
- η_v: the number of tree edges built by v in $T(v_0)$.

⋆ $c_v(s) \leq \alpha(\eta_v + 1) + \mathrm{Dist}(v_0) + n - 1.$

$c_{v_0}(s) = \alpha \cdot \eta_{v_0} + \mathrm{Dist}(v_0).$

- $c(s) = \sum_{v \in V(G_s) \setminus \{v_0\}} c_v(s) + c_{v_0}(s)$
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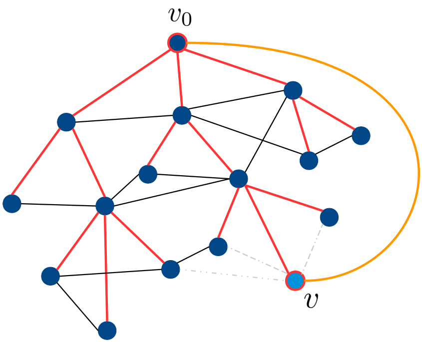

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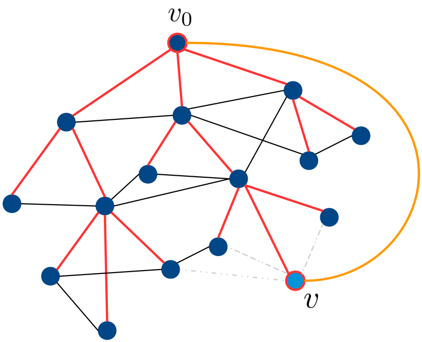

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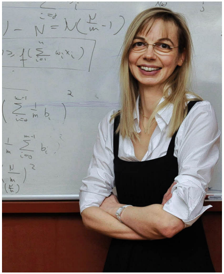
Price of Anarchy in Terms of Tree-Depth

Lemma 2 [Albers *et al.* @*SODA*'06]

If the shortest-path tree in an equilibrium graph G_s rooted at u has depth d, then PoA $\leq d + 1$.



- For some $u \in V$,

$$\begin{aligned}
\text{PoA} &\leq \frac{2\alpha(n-1) + n \cdot \text{Dist}(u) + (n-1)^2}{\alpha(n-1) + n(n-1)} \\
&\leq \frac{2\alpha(n-1) + n \cdot (n-1)d + (n-1)^2}{\alpha(n-1) + n(n-1)} \\
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&\leq \max\left\{\frac{2\alpha(n-1)}{\alpha(n-1)}, \frac{n(n-1)(d+1))}{n(n-1)}\right\} = \max\{2, d+1\}.
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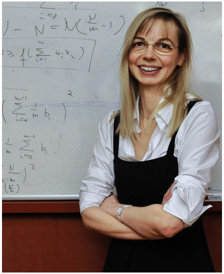

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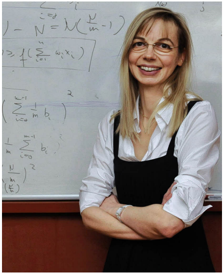

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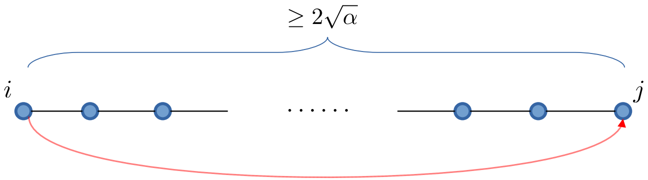
The General PoA Upper Bound

- If $\alpha > n^2$: the equilibrium graph is a tree.
- For $\alpha < n^2$: Note that $d_G(i,j) < 2\sqrt{\alpha}$.



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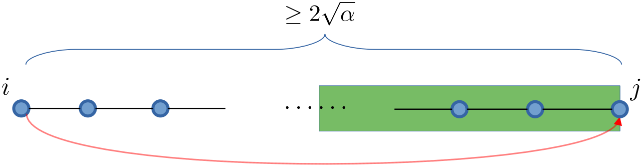

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Local Clustering of Agents

- $N_k(u)$: the set of vertices with distance $\leq k$ from u.

Lemma 3

For any equilibrium graph $G_\mathbf{s}$, $|N_2(u)| > \frac{n}{2\alpha}$ for every vertex u and $\alpha \geq 1$.

- Assume that $|\{v \in V(G_s) \mid d_{G_\mathbf{s}}(v, u) > 2\}| \geq \frac{n}{2}$.
  - Otherwise, $|N_2(u)| \geq n/2 \geq n/(2\alpha)$.

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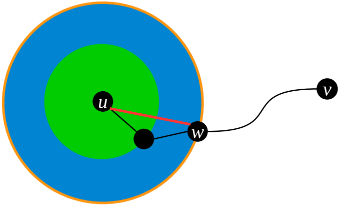
Proof of Lemma 3



- $S := \{v \in V \mid d_{G_s}(v, u) = 2\}$.
- For each v with $d_{G_s}(v, u) \geq 2$, pick any one of its shortest path to u and assign v to the only vertex (w) in this path that is in S.
- $|$ vertices assigned to $w \in S$ $| \leq \alpha$.
  - Otherwise, u could buy (u, w).
- $\therefore |S| \geq (n/2)/\alpha = n/(2\alpha)$.

: $\{v \in V \mid d_{G_s}(v, u) \leq 1\}$

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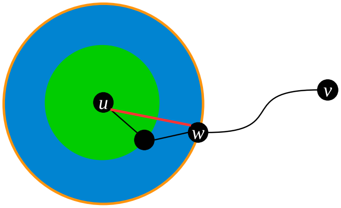

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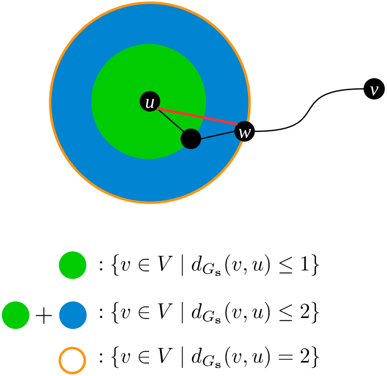

- $S := \{v \in V \mid d_{G_s}(v,u) = 2\}$.
- For each v with $d_{G_s}(v,u) \geq 2$, pick any one of its shortest path to u and assign v to the only vertex (w) in this path that is in S.
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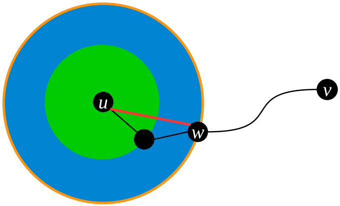

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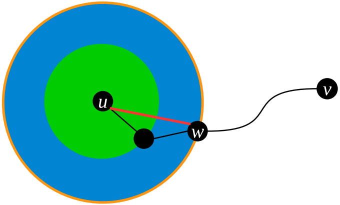

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For $\alpha < \sqrt{n/2}$, the PoA ≤ 6.

Key: *Show that $T(u)$ has depth ≤ 5 for any $u \in V(G_s)$.*

- Suppose that $\exists v \in V(G_s)$ s.t. $d_{G_s}(u, v) \geq 6$.
- v can buy $\{u, v\}$ to decrease its distance from all vertices in $N_2(u)$ by at least 1.
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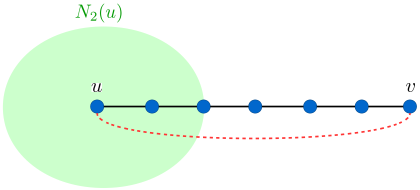

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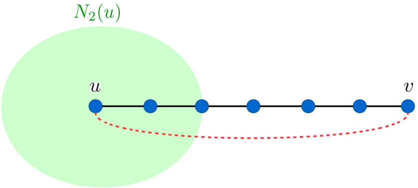

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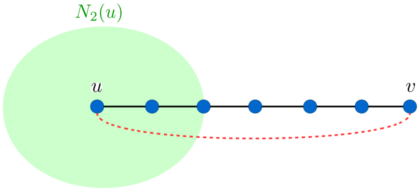

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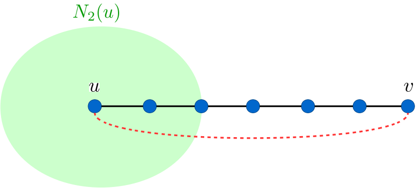

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- Δ: maximum vertex degree of G_s.
- $N_2(u) \leq 1 + \Delta + \Delta(\Delta - 1) = 1 + \Delta^2$ for an arbitrary u.
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- Let v be a vertex with degree Δ.
- Suppose that $\exists u \in V(G_s)$ s.t. $d_s(v, u) \geq 4$.
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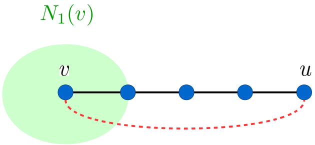

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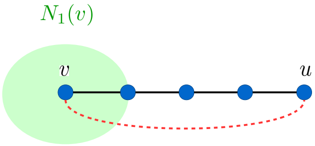

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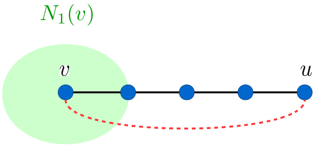

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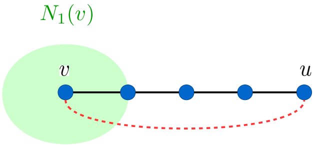

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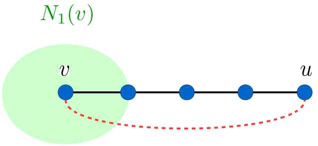


A brief summary

$\alpha =$	0 – 1	1 – 2	2 – $\sqrt[3]{n/2}$	$\sqrt[3]{n/2}$ – $\sqrt{n/2}$	$\sqrt{n/2}$ – $O(n^{1-\epsilon})$	$O(n^{1-\epsilon})$ – $65n$	$65n$ – $12n\lceil\lg n\rceil$	$12n\lceil\lg n\rceil$ – ∞
	1	$\frac{4}{3}$	≤ 4	≤ 6	$O(1)$	$2^{O(\sqrt{\lg n})}$	< 5	≤ 1.5

— : Fabrikant *et al.* 2003

— : Demaine *et al.* 2007

— : Albers *et al.* 2006

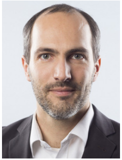
— : Mamageishvili, Mihalák & Müller 2013

A Quick View of the Social Network Creation Game

- D. Biló, T. Friedrich, P. Lenzner and S. Lowski: Selfish Creation of Social Networks. In *Proceedings of the Thirty-Fifth AAAI Conference on Artificial Intelligence* (*AAAI*'21).

HPI Hasso Plattner Institut
Digital Engineering · Universität Potsdam


Bilò Davide


Tobias Friedrich


Pascal Lenzner


Stefanie Lowski


Anna Melnichenko

Basic properties of real world social networks

- Small-world property: logarithmic diameter and average distances.
- Clustering: Two nodes with a common neighbor are neighbors w.h.p.
- Power-law degree distribution: Pr[a node has degree k] $\sim k^{-\beta}$, for $2 \leq \beta \leq 3$ (*scale-free*).

Pairwise Stability

Pairwise stable

$G = (V, E)$ is pairwise stable iff the following conditions hold:

- for every edge $uv \in E$, $c_u(G - uv) \geq c_u(G)$ and $c_v(G - uv) \geq c_v(G)$.
- for every *non-edge* $uv \notin E$, $c_u(G + uv) \geq c_u(G)$ and $c_v(G + uv) \geq c_v(G)$.

Monotone convex cost function

Monotonically increasing convex cost function

Let $\sigma : \mathbb{R} \mapsto \mathbb{R}$ be a monotonically increasing convex function such that $\sigma(0) = 0$. The cost of the edge uv in G is equal to:

$$c_{uv}(G) = \begin{cases} \sigma(d_{G-uv}(u,v)) & \text{if } d_{G-uv}(u,v) \neq \infty, \\ \sigma(n) & \text{otherwise.} \end{cases}$$

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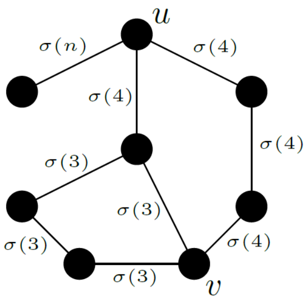
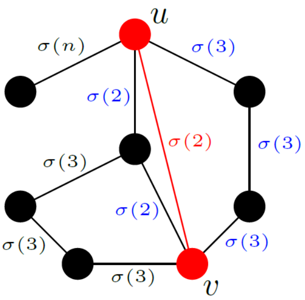
Changes after adding an edge





The cost of each agent

$$c_u(G) := \frac{1}{2} \sum_{v \in N_G(u)} c_{uv}(G) + \sum_{v \in V} d_G(u, v).$$

- $E = \{u, v \mid u, v \in V, u \in s_v, v \in s_u\}$.

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The Property of $\sigma(\cdot)$

Proposition

- Fix a positive real x.
- Let $x_1, \ldots, x_k$ with $0 \leq x_i \leq x$, be $k \geq 2$ positive reals.
- Let $\lambda_1, \ldots, \lambda_k$, with $\lambda_i \in [0, 1]$, such that $x = \sum_{i=1}^{k}(\lambda_i x_i)$.

Then

$$\sigma(x) \geq \sum_{i=1}^{k}(\lambda_i \sigma(x_i)).$$

- E.g., $\sigma(4) \geq 2\sigma(2)$, $\sigma(n) \geq 2\sigma(n/2)$.

⋆ Simulation uses $\sigma(x) := 2\log_2(n) \cdot x^{\alpha}$, for N agents and $\alpha \in \mathbb{R}$.

The Property of $\sigma(\cdot)$ (contd.)

- Indeed, let $r_i := \frac{x_i}{x} \in [0,1]$, i.e., $x_i = r_i x$.
- By convexity of σ, we have

$$\sigma(x_i) = \sigma((1-r_i)0 + r_i x) \leq (1-r_i)0 + r_i\sigma(x) = \frac{x_i}{x}\sigma(x).$$

Hence

$$\sum_{i=1}^{k}(\lambda_i\sigma(x_i)) \leq \frac{\sigma(x)}{x}\sum_{i=1}^{k}(\lambda_i x_i) = \sigma(x).$$

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- The number of expensive edges incident to any node is limited.

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In any pairwise stable network, any node has at most one incident edge of cost $\geq 2\sigma(2)$.

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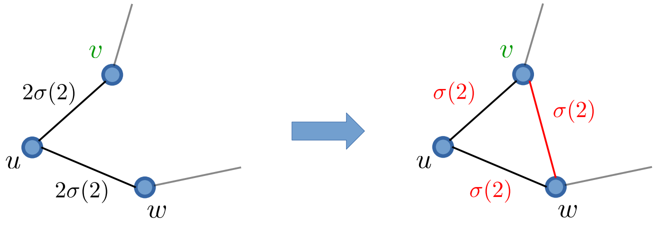

No more than three bridges

Proposition 3

Any pairwise stable network contains ≤ 3 bridges.

- v is a cut node: $G - v$ is disconnected.
- 2-connected graph: a connected graph with no cut node.

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graph G

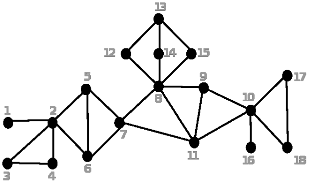
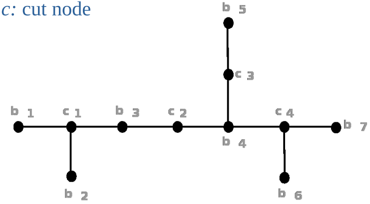


Figure from Wikipedia.

block-cut tree decomposition of G

tree nodes:
b: maximal 2-connected component
c: cut node



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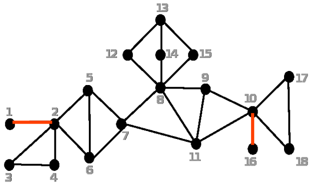

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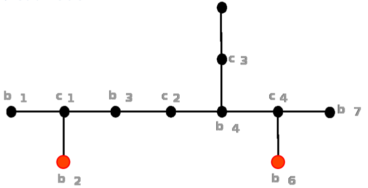

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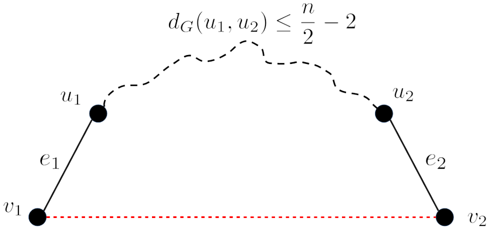
Sketch of the proof of Proposition 3

Claim: If there are ≥ 4 bridges in the graph, then there are two of them at a distance of $\leq n/2 - 2$.

- two bridges at distance $\leq n/2 - 2$ in $G \Rightarrow G$ is NOT pairwise stable.



$d_G(v_1, v_2) \leq \frac{n}{2}$

edge cost of v_1 : $\frac{\sigma(n)}{2}$

edge cost of v_1 after adding $v_1 v_2$:

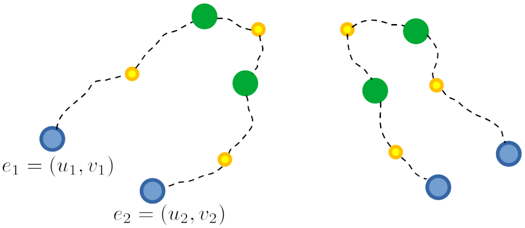
$\frac{1}{2}\sigma(\frac{n}{2}) \cdot 2 = \sigma(\frac{n}{2}) \leq \frac{\sigma(n)}{2}$.

distance cost of v_1 decreased by ≥ 1

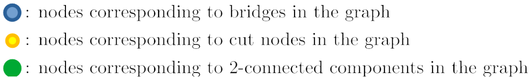
Sketch of the proof of Proposition 3 (contd.)

- Then we prove the claim.

four bridges and at least two bridge-bridge paths are node disjoint



a block-cut tree decomposition



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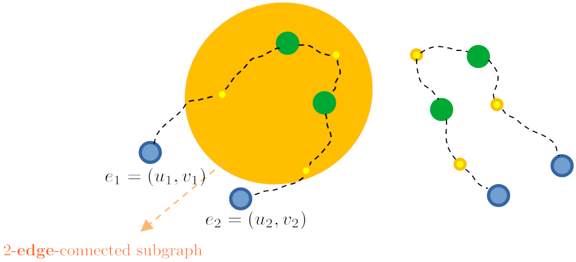

$$d(e_1, e_2) \le \frac{2}{3}\left(\frac{n}{2} - 2\right) \le \frac{n}{2} - 2.$$

A 2-edge-connected graph with n nodes has diameter bounded by $2n/3$. [Cacceta & Smyth 1992]

Sketch of the proof of Proposition 3 (contd.)

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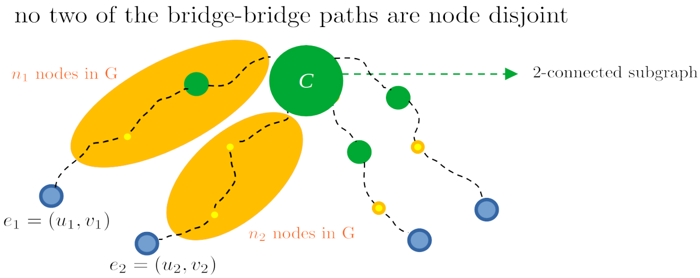

$d(e_1, e_2) \leq \left\lceil \frac{n_C - 1}{2} \right\rceil + \frac{2}{3}(n_1 - 1) + \frac{2}{3}(n_2 - 1).$

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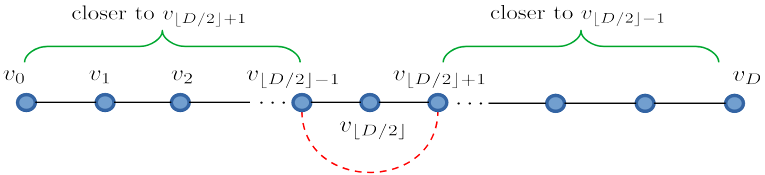
Bounded diameter in terms of $\sigma(2)$

- The upper bound on the diameter of any pairwise stable network only depends on the cost of edges closing a triangle.

Proposition 4

The diameter of any pairwise stable network is $\leq \sigma(2) + 2$.



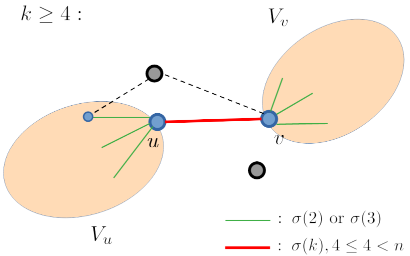
$\sigma(2)/2 - \lfloor D/2 \rfloor \geq 0.$

Cost for non-bridge edges

Proposition 5

In a pairwise stable network,

- for all $4 \leq k < n$, the cost of any k-edge is $\sigma(k) < n\sigma(2)$.
- if $\sigma(2) \leq \frac{1}{2}\sigma(3)$, then for all $3 \leq k < n$, the cost of any k-edge is $\sigma(k) \leq n\sigma(2)$.



V_u : nodes such that all shortest paths from v through uv

delete uv

$\Rightarrow$ the costs of "green" edges does not increase

the cost of u decreases by $\frac{1}{2}c_G(uv)$

distance cost of u increased by

$\leq |V_v|(d_{G-uv}(u, v) - 1) = |V_v|(k-1)$.

$$-\sigma(k)/2 + |V_v|(k-1) \geq 0$$
$$-\sigma(k)/2 + |V_u|(k-1) \geq 0$$
$$\Rightarrow \sigma(k) \leq (|V_u| + |V_v|)(k-1).$$

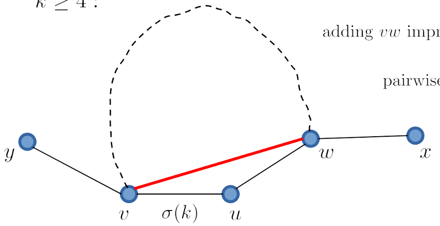
Cost for non-bridge edges

Proposition 5

In a pairwise stable network,

- for all $4 \leq k < n$, the cost of any k-edge is $\sigma(k) < n\sigma(2)$.
- if $\sigma(2) \leq \frac{1}{2}\sigma(3)$, then for all $3 \leq k < n$, the cost of any k-edge is $\sigma(k) \leq n\sigma(2)$.

$k \geq 4:$



adding vw improves the distance to $\geq \frac{k-2}{2}$ nodes in the cycle

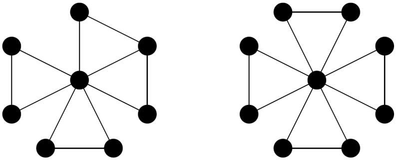
pairwise stable $\Rightarrow \frac{\sigma(2)}{2} - \frac{k-2}{2} - 1 \geq 0$

$\Rightarrow k \leq \sigma(2)$

The Social Optima

Theorem 1

- If $\sigma(2) < 2$, then K_n is the unique social optimum.
- If $\sigma(2) > 2$, then F_n is the unique social optimum.
- If $\sigma(2) = 2$, then any network of diameter 2 and containing only 2-edges is a social optimum.

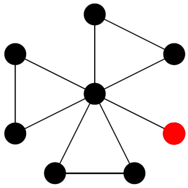


$\mathbf{F}_8$ fan graphs $\mathbf{F}_9$

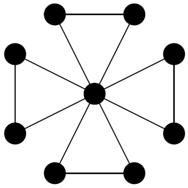
The Equilibrium Graphs

Theorem 2

- If $\sigma(2) < 2$, then K_n is the unique pairwise stable network.
- If $\sigma(2) \geq 2$, then F'_n is a pairwise stable network.



F'_8



$F'_9 = F_9$

modified fan graphs

The PoA Bound

Theorem 3 (The PoA Bound)

The PoA of Social Network Creation Game is

$$O\left(\min\{\sigma(2), n\} + \frac{\sigma(n)}{n \max\{\sigma(2), n\}}\right).$$

and

$$\Omega\left(\frac{\sigma(n)}{n \max\{\sigma(2), n\}}\right).$$

- Obviously, PoA $= 1$ when $\sigma(2) < 2$.

Sketch of the Proof of Theorem 3

- Social optimum F_n of cost $\Omega(n^2 + \sigma(2)n)$.
- Consider a pairwise stable network G.
  - Diameter of G: $\leq \sigma(2) + 2$ (Proposition 4).
  - The *distance cost* of $G \leq (\sigma(2) + 2) \cdot \binom{n}{2} \cdot 2$.
  - k_i: the number of k-edges in G.
  - The *edge cost* of G is at most

$$k_2 \cdot \sigma(2) + k_3 \cdot \sigma(3) + \sum_{i=3}^{n-1}(\sigma(i) \cdot k_i) + k_n\sigma(n) \leq 2\sigma(2)n^2 + k_n\sigma(n).$$

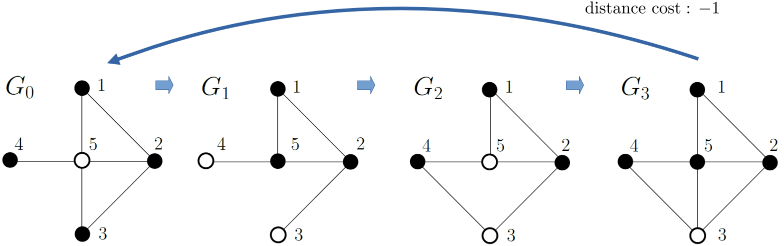
    - $\sigma(i) \leq n\sigma(2)$ for $4 \leq i < n$ (Proposition 5).
    - $k_3 \leq n^2/4$ [Mantel's Theorem, 1907].
    - $k_n \leq 3$ (Proposition 3).

One-Shot-Game ⇒ Dynamics

The SNCG does not have the *finite improvement property* (FIP).

Assume $2 < \sigma(2) < \frac{1}{2}\sigma(3) + 1$ and $\sigma(3) < \frac{1}{2}\sigma(5) + 2$.

$\frac{1}{2}\sigma(2) > 1$
distance cost : -1



$\frac{1}{2}\sigma(2) > 1.$
distance cost : -1

$2 \cdot \frac{1}{2}\sigma(3) - \frac{1}{2}\sigma(5) < 2.$
distance cost : -2

$3 \cdot \frac{1}{2}\sigma(2) - 2 \cdot \frac{1}{2}\sigma(3) \leq 0.$
$4 \cdot \frac{1}{2}\sigma(2) - 2 \cdot \frac{1}{2}\sigma(2) - \frac{1}{2}\sigma(3) < 1.$
distance cost : -1

Experiments

- Determine the edge cost function $\sigma(x)$.
  - Convex, monotone & $\sigma(0) = 0$.
  - The diameter at bounded by $\sigma(2) + 2$.
- Determine the initial network.
  - A cycle or a spanning tree.
- Activation scheme (updating steps).
  - Pick an agent (node) uniformly at random.
  - Let the agent perform the best possible edge addition/deletion.

Experiments (contd.)

- $\sigma(x) := \beta x^\alpha$ or $\sigma(x) := 2\log_2(n) \cdot x^\alpha$, for n agents and $\alpha, \beta \in \mathbb{R}$.
    - In line with observed results in real-world networks [Barabási 2016].
- $n = 1000$ (or 3000) for 20 runs. Each run, which contains updating steps, starts from a sparse initial cycle or a random spanning tree.
    - In each step, one agent is activated uniformly at random and then it performs the best possible edge addition (jointly with other endpoint which agrees) or edge deletion.
    - If no such move exists then the agent is marked, otherwise the network is updated. The process stops when all agents are marked.

Remarks on the experiments

- Additional experiments starting with Erdös-Renyi random networks.
  - The network initialization does not matter as long as the networks are sparse and average distances are large.
  - However, starting from a "star" network yields drastically different results.
  - Starting from a fan graph $\Rightarrow$ the algorithm stops immediately since it is pairwise stable.

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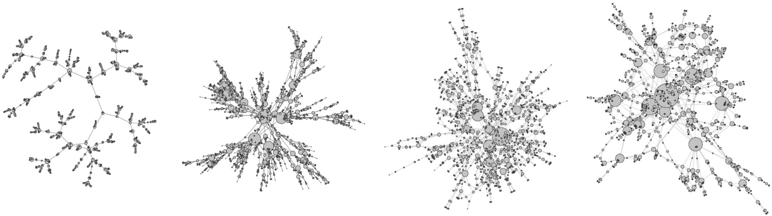
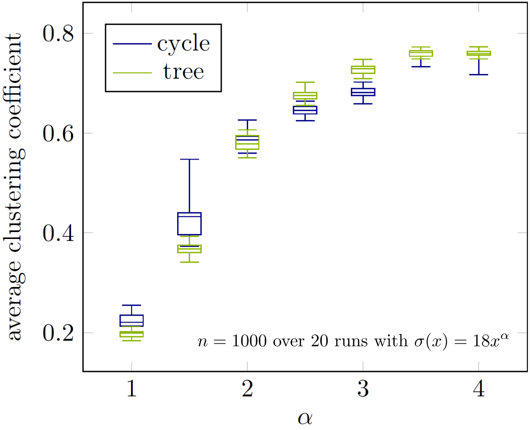
Simulation of the Dynamics



- Starting from a random spanning tree with $n = 1000$ and $\alpha = 3$.
- Left to right: current network after 1,000 steps each.

Clustering coefficient of pairwise stable networks

- $CC(v) := \frac{\Delta(v)}{\deg(v)(\deg(v)-1)}$, where $\Delta(v)$: the number of triangles containing v.
- $CC(G) = \frac{1}{n}\sum_{v \in V} CC(v)$.



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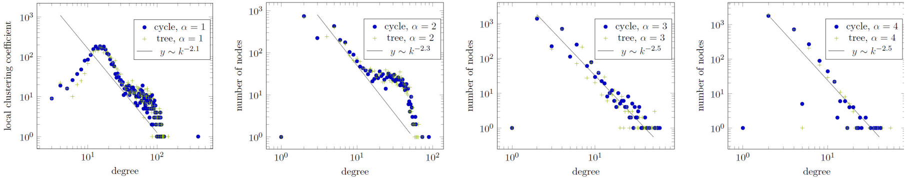

$n = 3000$
$\sigma(x) = 22x^{\alpha}$

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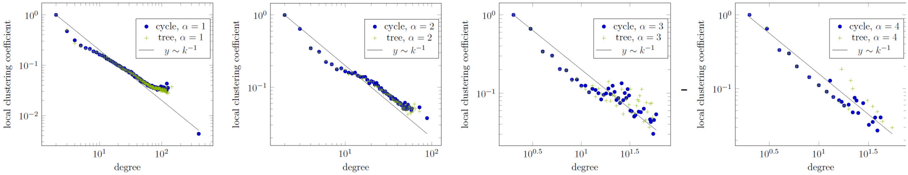

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Comparisons with the real-world social networks

	SNCG, $\alpha = 2$	SNCG, $\alpha = 3$	ego Facebook [34]	ADVOGATO [44]	HAMSTERSTER [44]
$\|V\|$	3000	3000	4039	2280	1348
$\|E\|$	18059	6019	88234	5251	6642
Diameter	8	11	8	11	6
avg distance	3.69	5.17	3.69	3.85	3.2
max degree	72	55	1045	148	273
avg degree	12	4.013	43.7	4.61	9.85
avg CC	0.415	0.67	0.617	0.2868	0.54

Discussion.

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