No-Regret Online Learning Algorithms

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Lecture Notes

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Credits for the resource

The slides are based on the lectures of Prof. Luca Trevisan:
https://lucatrevisan.github.io/40391/index.html

the lectures of Prof. Shipra Agrawal:
https://ieor8100.github.io/mab/

the lectures of Prof. Francesco Orabona:
https://parameterfree.com/lecture-notes-on-online-learning/

and also Elad Hazan's textbook:
Introduction to Online Convex Optimization, 2nd Edition.

Outline

1. Introduction
2. Gradient Descent for Online Convex Optimization (GD)
3. Multiplicative Weight Update (MWU)
4. Follow The Leader (FTL)
5. Follow The Regularized Leader (FTRL)
   - MWU Revisited
   - FTRL with 2-norm regularizer
6. Multi-Armed Bandit (MAB)
   - Greedy Algorithms
   - Upper Confidence Bound (UCB)
   - Time-Decay ϵ-Greedy

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Online Convex Optimization

Goal: Design an algorithm such that

- At discrete time steps $t = 1, 2, \ldots$, output $\boldsymbol{x}_t \in \mathcal{K}$, for each t.
  - $\mathcal{K}$: a convex set of feasible solutions.
- After $\boldsymbol{x}_t$ is generated, a convex cost function $f_t : \mathcal{K} \mapsto \mathbb{R}$ is revealed.
- Then the algorithm suffers the loss $f_t(\boldsymbol{x}_t)$.

And we want to minimize the cost.

The difficulty

- The cost functions f_t is unknown before t.
- $f_1, f_2, \ldots, f_t, \ldots$ are not necessarily fixed.
  - Can be generated dynamically by an adversary.

What's the regret?

- The offline optimum: After T steps,

$$\min_{\boldsymbol{x} \in \mathcal{K}} \sum_{t=1}^{T} f_t(\boldsymbol{x}).$$

- The regret after T steps:

$$\text{regret}_T = \sum_{t=1}^{T} f_t(\boldsymbol{x}_t) - \min_{\boldsymbol{x} \in \mathcal{K}} \sum_{t=1}^{T} f_t(\boldsymbol{x}).$$

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- The rescue: $\text{regret}_T \leq o(T)$. $\Rightarrow$ **No-Regret** in average when $T \to \infty$.
  - For example, $\text{regret}_T / T = \frac{\sqrt{T}}{T} \to 0$ when $T \to \infty$.

Prerequisites (1/5)

Diameter

Let $\mathcal{K} \subseteq \mathbb{R}^d$ be a bounded convex and closed set in Euclidean space. We denote by D an upper bound on the diameter of $\mathcal{K}$:

$$\forall \boldsymbol{x}, \boldsymbol{y} \in \mathcal{K}, ||\boldsymbol{x} - \boldsymbol{y}|| \leq D.$$

Convex set

A set $\mathcal{K}$ is convex if for any $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{K}$, we have

$$\forall \alpha \in [0, 1], \alpha \boldsymbol{x} + (1 - \alpha) \boldsymbol{y} \in \mathcal{K}.$$

Prerequisites (2/5)

Convex function

A function $f : \mathcal{K} \mapsto \mathbb{R}$ is convex if for any $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{K}$,

$$\forall \alpha \in [0,1], f((1-\alpha)\boldsymbol{x} + \alpha \boldsymbol{y}) \leq (1-\alpha) f(\boldsymbol{x}) + \alpha f(\boldsymbol{y}).$$

Equivalently, if f is differentiable (i.e., $\nabla f(\boldsymbol{x})$ exists for all $\boldsymbol{x} \in \mathcal{K}$), then f is convex if and only if for all $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{K}$,

$$f(\boldsymbol{y}) \geq f(\boldsymbol{x}) + \nabla f(\boldsymbol{x})^\top (\boldsymbol{y} - \boldsymbol{x}).$$

Prerequisites (3/5)

Theorem [Rockafellar 1970]

Suppose that $f : \mathcal{K} \mapsto \mathbb{R}$ is a convex function and let $x \in \text{int dom}(f)$. If f is differentiable at $\boldsymbol{x}$, then for all $\boldsymbol{y} \in \mathbb{R}^d$,

$$f(\boldsymbol{y}) \geq f(\boldsymbol{x}) + \langle \nabla f(\boldsymbol{x}), \boldsymbol{y} - \boldsymbol{x} \rangle.$$

Subgradient

For a function $f : \mathbb{R}^d \mapsto \mathbb{R}$, $\boldsymbol{g} \in \mathbb{R}^d$ is a subgradient of f at $x \in \mathbb{R}^d$ if for all $\boldsymbol{y} \in \mathbb{R}^d$,

$$f(\boldsymbol{y}) \geq f(\boldsymbol{x}) + \langle \boldsymbol{g}, \boldsymbol{y} - \boldsymbol{x} \rangle.$$

Prerequisites (4/5)

Projection

The closest point of $\boldsymbol{y}$ in a convex set $\mathcal{K}$ in terms of norm $||\cdot||$:

$$\Pi_{\mathcal{K}}(\boldsymbol{y}) := \arg\min_{\boldsymbol{x}\in\mathcal{K}} ||\boldsymbol{x}-\boldsymbol{y}||.$$

Pythagoras Theorem

Let $\mathcal{K} \subseteq \mathbb{R}^d$ be a convex set, $\boldsymbol{y} \in \mathbb{R}^d$ and $\boldsymbol{x} = \Pi_{\mathcal{K}}(\boldsymbol{y})$. Then for any $\boldsymbol{z} \in \mathcal{K}$, we have

$$||\boldsymbol{y}-\boldsymbol{z}|| \geq ||\boldsymbol{x}-\boldsymbol{z}||.$$

Prerequisites (5/5)

Minimum vs. zero gradient

$$\nabla f(\boldsymbol{x}) = 0 \text{ iff } \boldsymbol{x} \in \arg\min_{\boldsymbol{x}\in\mathbb{R}^d}\{f(\boldsymbol{x})\}.$$

Karush-Kuhn-Tucker (KKT) Theorem

Let $\mathcal{K} \subseteq \mathbb{R}^d$ be a convex set, $\boldsymbol{x}^* \in \arg\min_{\boldsymbol{x}\in\mathcal{K}} f(\boldsymbol{x})$. Then for any $\boldsymbol{y} \in \mathcal{K}$ we have

$$\nabla f(\boldsymbol{x}^*)^\top(\boldsymbol{y} - \boldsymbol{x}^*) \geq 0.$$

Convex losses to linear losses

- We have the convex loss function $f_t(\boldsymbol{x}_t)$ at time t.
- Say we have subgradients $\boldsymbol{g}_t$ for each $\boldsymbol{x}_t$.
- $f(\boldsymbol{x}_t) - f(\boldsymbol{u}) \leq \langle \boldsymbol{g}, \boldsymbol{x_t} - \boldsymbol{u} \rangle$ for each $\boldsymbol{u} \in \mathbb{R}^d$.

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- $f(\boldsymbol{x}_t) - f(\boldsymbol{u}) \leq \langle \boldsymbol{g}, \boldsymbol{x_t} - \boldsymbol{u} \rangle$ for each $\boldsymbol{u} \in \mathbb{R}^d$.
- Hence, if we define $\tilde{f}_t(\boldsymbol{x}) := \langle \boldsymbol{g}_t, \boldsymbol{x} \rangle$, then for any $\boldsymbol{u} \in \mathbb{R}^d$,

$$\sum_{t=1}^{T} f_t(\boldsymbol{x}_t) - f(\boldsymbol{u}) \leq \sum_{t=1}^{T} \langle \boldsymbol{g}, \boldsymbol{x_t} - \boldsymbol{u} \rangle = \sum_{t=1}^{T} \tilde{f}_t(\boldsymbol{x}_t) - \tilde{f}(\boldsymbol{u}).$$

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OCO $\rightarrow$ OLO.

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Online Gradient Descent (GD)

1. **Input:** convex set $\mathcal{K}$, T, $\boldsymbol{x}_1 \in \mathcal{K}$, step size $\{\eta_t\}$.
2. **for** $t \leftarrow 1$ to T **do**:
   1. Play $\boldsymbol{x}_t$ and observe cost $f_t(\boldsymbol{x}_t)$.
   2. Update and Project:

$$
\begin{aligned}
\boldsymbol{y}_{t+1} &= \boldsymbol{x}_t - \eta_t \nabla f_t(\boldsymbol{x}_t) \\
\boldsymbol{x}_{t+1} &= \Pi_{\mathcal{K}}(\boldsymbol{y}_{t+1})
\end{aligned}
$$

3. **end for**

GD for online convex optimization is of no-regret

Theorem A

Online gradient descent with step size $\{\eta_t = \frac{D}{G\sqrt{t}}, t \in [T]\}$ guarantees the following for all $T \geq 1$:

$$\text{regret}_T = \sum_{t=1}^{T} f_t(\boldsymbol{x}_t) - \min_{\boldsymbol{x}^* \in \mathcal{K}} \sum_{t=1}^{T} f_t(\boldsymbol{x}^*) \leq \frac{3}{2} GD\sqrt{T}.$$

Proof of Theorem A (1/3)

- Let $\boldsymbol{x}^* \in \arg\min_{\boldsymbol{x} \in \mathcal{K}} \sum_{t=1}^{T} f_t(\boldsymbol{x})$.
- Since f_t is convex, we have

$$f_t(\boldsymbol{x}_t) - f_t(\boldsymbol{x}^*) \leq (\nabla f_t(\boldsymbol{x}_t))^\top (\boldsymbol{x}_t - \boldsymbol{x}^*).$$

- By the updating rule for $\boldsymbol{x}_{t+1}$ and the Pythagorean theorem, we have

$$||\boldsymbol{x}_{t+1} - \boldsymbol{x}^*||^2 = ||\Pi_{\mathcal{K}}(\boldsymbol{x}_t - \eta_t \nabla f_t(\boldsymbol{x}_t)) - \boldsymbol{x}^*||^2 \leq ||\boldsymbol{x}_t - \eta_t \nabla f_t(\boldsymbol{x}_t) - \boldsymbol{x}^*||^2.$$

Proof of Theorem A (2/3)

- Hence

$$||\boldsymbol{x}_{t+1} - \boldsymbol{x}^*||^2 \leq ||\boldsymbol{x}_t - \boldsymbol{x}^*||^2 + \eta_t^2 ||\nabla f_t(\boldsymbol{x}_t)||^2 - 2\eta_t (\nabla f_t(\boldsymbol{x}_t))^\top (\boldsymbol{x}_t - \boldsymbol{x}^*)$$

$$2(\nabla f_t(\boldsymbol{x}_t))^\top (\boldsymbol{x}_t - \boldsymbol{x}^*) \leq \frac{||\boldsymbol{x}_t - \boldsymbol{x}^*||^2 - ||\boldsymbol{x}_{t+1} - \boldsymbol{x}^*||^2}{\eta_t} + \eta_t G^2.$$

- Summing above inequality from $t = 1$ to T and setting $\eta_t = \frac{D}{G\sqrt{t}}$ and $\frac{1}{\eta_0} := 0$ we have :

Proof of Theorem A (3/3)

$$
\begin{aligned}
2\left(\sum_{t=1}^{T} f_t(\boldsymbol{x}_t) - f_t(\boldsymbol{x}^*)\right) &\leq 2\sum_{t=1}^{T} \nabla f_t(\boldsymbol{x}_t))^\top(\boldsymbol{x}_t - \boldsymbol{x}^*) \\
&\leq \sum_{t=1}^{T} \frac{||\boldsymbol{x}_t - \boldsymbol{x}^*||^2 - ||\boldsymbol{x}_{t+1} - \boldsymbol{x}^*||^2}{\eta_t} + G^2 \sum_{t=1}^{T} \eta_t \\
&\leq \sum_{t=1}^{T} ||\boldsymbol{x}_t - \boldsymbol{x}^*||^2 \left(\frac{1}{\eta_t} - \frac{1}{\eta_{t-1}}\right) + G^2 \sum_{t=1}^{T} \eta_t \\
&\leq D^2 \sum_{t=1}^{T} \left(\frac{1}{\eta_t} - \frac{1}{\eta_{t-1}}\right) + G^2 \sum_{t=1}^{T} \eta_t \\
&\leq D^2 \frac{1}{\eta_T} + G^2 \sum_{t=1}^{T} \eta_t \\
&\leq 3DG\sqrt{T}.
\end{aligned}
$$

The Lower Bound

Theorem B

Let $\mathcal{K} = \{\boldsymbol{x} \in \mathbb{R}^d : ||\boldsymbol{x}||_\infty \leq r\}$ be a convex subset of $\mathbb{R}^d$. Let A be any algorithm for Online Convex Optimization on $\mathcal{K}$. Then for any $T \geq 1$, there exists a sequence of vectors $\boldsymbol{g}_1, \ldots, \boldsymbol{g}_T$ with $||\boldsymbol{g}_t||_2 \leq L$ and $\boldsymbol{u} \in \mathcal{K}$ such that the regret of A satisfies

$$\text{regret}_T(\boldsymbol{u}) = \sum_{t=1}^{T} \langle \boldsymbol{g}_t, \boldsymbol{x}_t \rangle - \sum_{t=1}^{T} \langle \boldsymbol{g}_t, \boldsymbol{u} \rangle \geq \frac{\sqrt{2}LD\sqrt{T}}{4}.$$

- The diameter D of $\mathcal{K}$ is at most $\sqrt{\sum_{i=1}^{d}(2r)^2} \leq 2r\sqrt{d}$.
- $||\boldsymbol{x}||_\infty \leq r \Leftrightarrow |\boldsymbol{x}(i)| \leq r$ for each $i \in [n]$.

Proof of Theorem B (1/2)

- The approach:

  For any random variable $\boldsymbol{z}$ with domain $\mathcal{V}$ and any function f,

$$\sup_{\boldsymbol{x} \in V} f(\boldsymbol{x}) \geq E[f(\boldsymbol{z})].$$

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- Let $\boldsymbol{z} := \frac{\boldsymbol{v}-\boldsymbol{w}}{||\boldsymbol{v}-\boldsymbol{w}||} \Rightarrow \langle \boldsymbol{z}, \boldsymbol{v} - \boldsymbol{w}\rangle = D$.
- Let $\epsilon_1, \epsilon_2, \ldots, \epsilon_T$ be i.i.d. random variables such that $\Pr[\epsilon_t = 1] = \Pr[\epsilon_t = -1] = 1/2$ for each t.

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- We choose the losses $\boldsymbol{g}_t = L\epsilon_t \boldsymbol{z}$.

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- We choose the losses $\boldsymbol{g}_t = L\epsilon_t \boldsymbol{z}$.
  - The cost at t: $\langle L\epsilon_t \boldsymbol{z}, \boldsymbol{x}_t \rangle$.
  - $||g_t|| = \sqrt{L^2 \epsilon_t^2} \cdot ||\boldsymbol{z}|| \leq L$.

Proof of Theorem B (2/2)

$$\begin{aligned}
\sup_{\boldsymbol{g}_1,\ldots,\boldsymbol{g}_T} \text{regret}_T &\geq E\left[\sum_{t=1}^{T} L\epsilon_t \langle \boldsymbol{z}, \boldsymbol{x}_t \rangle - \min_{\boldsymbol{u}\in\mathcal{K}} \sum_{t=1}^{T} L\epsilon_t \langle \boldsymbol{z}, \boldsymbol{u} \rangle\right] \\
&= E\left[-\min_{\boldsymbol{u}\in\mathcal{K}} \sum_{t=1}^{T} L\epsilon_t \langle \boldsymbol{z}, \boldsymbol{u} \rangle\right] = E\left[\max_{\boldsymbol{u}\in\mathcal{K}} \sum_{t=1}^{T} L\epsilon_t \langle \boldsymbol{z}, \boldsymbol{u} \rangle\right] \\
&\geq E\left[\max_{\boldsymbol{u}\in\{\boldsymbol{v},\boldsymbol{w}\}} \sum_{t=1}^{T} L\epsilon_t \langle \boldsymbol{z}, \boldsymbol{u} \rangle\right] \\
&= E\left[\frac{1}{2}\sum_{t=1}^{T} L\epsilon_t \langle \boldsymbol{z}, \boldsymbol{v}+\boldsymbol{w} \rangle + \frac{1}{2}\left|\sum_{t=1}^{T} L\epsilon_t \langle \boldsymbol{z}, \boldsymbol{v}-\boldsymbol{w} \rangle\right|\right] \\
&\geq \frac{L}{2} E\left[\left|\sum_{t=1}^{T} L\epsilon_t \langle \boldsymbol{z}, \boldsymbol{v}-\boldsymbol{w} \rangle\right|\right] = \frac{LD}{2} E\left[\left|\sum_{t=1}^{T} \epsilon_t\right|\right] \\
&\geq \frac{\sqrt{2}LD\sqrt{T}}{4}. \quad \text{(by Khintchine inequality)}
\end{aligned}$$

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- The idea: at each time step, decide the probability distribution (i.e., weights) of the experts to follow their advice.
  - $\boldsymbol{x}_t = (\boldsymbol{x}_t(1), \boldsymbol{x}_t(2), \ldots, \boldsymbol{x}_t(n))$, where $\boldsymbol{x}_t(i) \in [0, 1]$ and $\sum_i \boldsymbol{x}_t(i) = 1$.

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- The loss of following expert i at time t: $\ell_t(i)$.
- The expected loss of the algorithm at time t:

$$\langle \boldsymbol{x}_t, \ell_t \rangle = \sum_{i=1}^{n} \boldsymbol{x}_t(i)\ell_t(i).$$

The regret of listening to the experts...

$$\text{regret}_T^* = \sum_{t=1}^{T} \langle \boldsymbol{x}_t, \boldsymbol{\ell}_t \rangle - \min_i \sum_{t=1}^{T} \ell_t(i).$$

- The set of feasible solutions $K = \Delta \subseteq \mathbb{R}^n$, probability distributions over $\{1, \ldots, n\}$.
- $f_t(\boldsymbol{x}) = \sum_i \boldsymbol{x}(i)\ell_t(i)$: linear function.

⋆ Assume that $|\ell_t(i)| \leq 1$ for all t and i.

The MWU Algorithm

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Multiplicative Weight Update (MWU)

- Maintain a vector of weights $\boldsymbol{w}_t = (\boldsymbol{w}_t(1), \ldots, \boldsymbol{w}_t(n))$ where $\boldsymbol{w}_1 := (1, 1, \ldots, 1)$.
- Update the weights at time t by
  - $\boldsymbol{w}_t(i) := \boldsymbol{w}_{t-1}(i) \cdot e^{-\beta \boldsymbol{\ell}_{t-1}(i)}$.
  - $\boldsymbol{x}_t := \frac{\boldsymbol{w}_t(i)}{\sum_{j=1}^{n} \boldsymbol{w}_t(j)}$.

β: a parameter which will be optimized later.

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β: a parameter which will be optimized later.

The weight of expert i at time t: $e^{-\beta \sum_{k=1}^{t-1} \ell_k(i)}$.

MWU is of no-regret

Theorem 1 (MWU is of no-regret)

Assume that $|\ell_t(i)| \leq 1$ for all t and i. For $\beta \in (0, 1/2)$, the regret of MWU after T steps is bounded as

$$\text{regret}_T^* \leq \beta \sum_{t=1}^{T} \sum_{i=1}^{n} \boldsymbol{x}_t(i) \ell_t^2(i) + \frac{\ln n}{\beta} \leq \beta T + \frac{\ln n}{\beta}.$$

In particular, if $T > 4 \ln n$, then

$$\text{regret}_T^* \leq 2\sqrt{T \ln n}$$

by setting $\beta = \sqrt{\frac{\ln n}{T}}$.

Proof of Theorem 1

Let $W_t := \sum_{i=1}^{n} \boldsymbol{w}_t(i)$.

The idea:

- If the algorithm incurs a large loss after T steps, then W_{T+1} is small.
- And, if W_{T+1} is small, then even the best expert performs quite badly.

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- If the algorithm incurs a large loss after T steps, then W_{T+1} is small.
- And, if W_{T+1} is small, then even the best expert performs quite badly.

Let $L^* := \min_i \sum_{t=1}^{T} \ell_t(i)$.

The proof (contd.)

Lemma 1 (W_{T+1} is SMALL $\Rightarrow$ L^* is LARGE)

$W_{T+1} \geq e^{-\beta L^*}$.

Proof.

Let $j = \arg\min L^* = \arg\min_i \sum_{t=1}^{T} \ell_t(i)$.

$$W_{T+1} = \sum_{i=1}^{n} e^{-\beta \sum_{t=1}^{T} \ell_t(i)} \geq e^{-\beta \sum_{t=1}^{T} \ell_t(j)} = e^{-\beta L^*}.$$

□

The proof (contd.)

Lemma 2 (MWU brings large loss $\Rightarrow W_{T+1}$ is SMALL)

$$W_{T+1} \leq n \prod_{t=1}^{n} (1 - \beta \langle \boldsymbol{x}_t, \boldsymbol{\ell}_t \rangle + \beta^2 \langle \boldsymbol{x}_t, \boldsymbol{\ell}_t^2 \rangle),$$

Proof.

Note: $W_1 = n$.

$$\frac{W_{t+1}}{W_t} = \sum_{i=1}^{n} \frac{\boldsymbol{w}_{t+1}(i)}{W_t} = \sum_{i=1}^{n} \frac{\boldsymbol{w}_t(i) \cdot e^{-\beta \boldsymbol{\ell}_t(i)}}{W_t}$$

The proof (contd.)

Lemma 2 (MWU brings large loss ⇒ W_{T+1} is SMALL)

$$W_{T+1} \leq n \prod_{t=1}^{n} (1 - \beta \langle \boldsymbol{x}_t, \boldsymbol{\ell}_t \rangle + \beta^2 \langle \boldsymbol{x}_t, \boldsymbol{\ell}_t^2 \rangle),$$

Proof.

Note: $W_1 = n$.

$$\frac{W_{t+1}}{W_t} = \sum_{i=1}^{n} \frac{\boldsymbol{w}_{t+1}(i)}{W_t} = \sum_{i=1}^{n} \frac{\boldsymbol{w}_t(i) \cdot e^{-\beta \boldsymbol{\ell}_t(i)}}{W_t} = \sum_{i=1}^{n} \boldsymbol{x}_t(i) \cdot e^{-\beta \boldsymbol{\ell}_t(i)}$$

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□

The proof (contd.)

Lemma 2 (MWU brings large loss $\Rightarrow W_{T+1}$ is SMALL)

$$W_{T+1} \le n \prod_{t=1}^{n} e^{-\beta\langle x_t, \ell_t\rangle + \beta^2\langle x_t, \ell_t^2\rangle}.$$

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\end{aligned}$$

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The proof (contd.)

Hence

$$\ln W_{T+1} \leq \ln n - \left(\sum_{i=1}^{T} \beta \langle \boldsymbol{\ell}_t, \boldsymbol{x}_t \rangle \right) + \left(\sum_{i=1}^{T} \beta^2 \langle \boldsymbol{\ell}_t^2, \boldsymbol{x}_t \rangle \right)$$

and $\ln W_{T+1} \geq -\beta L^*$.

The proof (contd.)

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Thus,

$$\left(\sum_{t=1}^{T} \langle \boldsymbol{\ell}_t, \mathbf{x}_t \rangle \right) - L^* \le \frac{\ln n}{\beta} + \beta \sum_{t=1}^{T} \langle \boldsymbol{\ell}_t^2, \mathbf{x}_t \rangle.$$

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Take $\beta = \sqrt{\frac{\ln n}{T}}$, we have $\text{regret}_T \leq 2\sqrt{T \ln n}$.

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Why so complicated?

- How about just *following the one with best performance*?

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- First, we assume to make no assumptions on $\mathcal{K}$ and $\{f_t : L \mapsto \mathbb{R}\}$.
- At time t, we are given previous cost functions $f_1, \ldots, f_{t-1}$, and then give the solution

$$\boldsymbol{x}_t := \arg\min_{\boldsymbol{x} \in \mathcal{K}} \sum_{k=1}^{t-1} f_k(\boldsymbol{x}).$$

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- It seems reasonable and makes sense, doesn't it?

FTL leads to "overfitting"

t:	1
$\boldsymbol{x}_t$:	$(0.5, 0.5)$
ℓ_t:	$(0, 0.5)$
$f_t(\boldsymbol{x}_t)$:	0.25
$\arg\min_{\boldsymbol{x}} \sum_{k=1}^{t} f_k(\boldsymbol{x})$:	$(1, 0)$

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t:	1	2
$\boldsymbol{x}_t$:	$(0.5, 0.5)$	$(1, 0)$
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FTL leads to "overfitting"

t:	1	2	3
$\boldsymbol{x}_t$:	$(0.5, 0.5)$	$(1, 0)$	$(0, 1)$
ℓ_t:	$(0, 0.5)$	$(1, 0)$	$(0, 1)$
$f_t(\boldsymbol{x}_t)$:	0.25	1	1
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FTL leads to "overfitting"

t:	1	2	3	4
$\boldsymbol{x}_t$:	$(0.5, 0.5)$	$(1, 0)$	$(0, 1)$	$(1, 0)$
ℓ_t:	$(0, 0.5)$	$(1, 0)$	$(0, 1)$	$(1, 0)$
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FTL leads to "overfitting"

t:	1	2	3	4	5
$\boldsymbol{x}_t$:	$(0.5, 0.5)$	$(1, 0)$	$(0, 1)$	$(1, 0)$	$(0, 1)$
ℓ_t:	$(0, 0.5)$	$(1, 0)$	$(0, 1)$	$(1, 0)$	$(0, 1)$
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optimum loss: $\approx T/2$.

FTL's loss: $\approx T$.

regret: $\approx T/2$ (linear).

Analysis of FTL

Theorem 2 (Analysis of FTL)

For any sequence of cost functions $f_1, \ldots, f_t$ and any number of time steps T, the FTL algorithm satisfies

$$\text{regret}_T \leq \sum_{t=1}^{T} (f_t(\boldsymbol{x}_t) - f_t(\boldsymbol{x}_{t+1})).$$

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Implication: If $f_t(\cdot)$ is Lipschitz w.r.t. to some distance function $||\cdot||$, then $\boldsymbol{x}_t$ and $\boldsymbol{x}_{t+1}$ are close $\Rightarrow ||f_t(\boldsymbol{x}_t) - f_t(\boldsymbol{x}_{t+1})||$ can't be too large.

Modify FTL: $\boldsymbol{x}_t$'s shouldn't change too much from step by step.

Proof of Theorem 2

Recall that

$$\text{regret}_T = \sum_{t=1}^{T} f_t(\boldsymbol{x}_t) - \min_{\boldsymbol{x}\in\mathcal{K}} \sum_{t=1}^{T} f_t(\boldsymbol{x})$$

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The theorem $\Leftrightarrow \sum_{t=1}^{T} f_t(\mathbf{x}_{t+1}) \leq \min_{\mathbf{x}\in\mathcal{K}} \sum_{t=1}^{T} f_t(\mathbf{x})$.

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where

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Introducing REGULARIZATION

- You might have already been using regularization for quite a long time.

Introducing REGULARIZATION

```
from keras import regularizers
model.add(Dense(64, input_dim=64,
                kernel_regularizer=regularizers.l2(0.01)
```

Introducing REGULARIZATION

```
# l1 data (only 5 informative features)
X_1, y_1 = datasets.make_classification(n_samples=n_samples,
                                        n_features=n_features, n_informative=5,
                                        random_state=1)

# l2 data: non sparse, but less features
y_2 = np.sign(.5 - rnd.rand(n_samples))
X_2 = rnd.randn(n_samples, n_features // 5) + y_2[:, np.newaxis]
X_2 += 5 * rnd.randn(n_samples, n_features // 5)

clf_sets = [(LinearSVC(penalty='l1', loss='squared_hinge', dual=False,
                       tol=1e-3),
             np.logspace(-2.3, -1.3, 10), X_1, y_1),
            (LinearSVC(penalty='l2', loss='squared_hinge', dual=True),
             np.logspace(-4.5, -2, 10), X_2, y_2)]
```

The regularizer

At each step, we compute the solution

$$\boldsymbol{x}_t := \arg\min_{\boldsymbol{x}\in\mathcal{K}} \left(R(\boldsymbol{x}) + \sum_{k=1}^{t-1} f_k(\boldsymbol{x}) \right).$$

This is called Follow the Regularized Leader (FTRL).

In short,

$$\text{FTRL} = \text{FTL} + \text{Regularizer}.$$

Analysis of FTRL

Theorem 3 (Analysis of FTRL)

For

- every sequence of cost function $\{f_t(\cdot)\}_{t\geq 1}$ and
- every regularizer function $R(\cdot)$,

for every $\boldsymbol{x}$, the regret with respect to $\boldsymbol{x}$ after T steps of the FTRL algorithm is bounded as

$$\mathrm{regret}_T(\boldsymbol{x}) \leq \left(\sum_{t=1}^{T} f_t(\boldsymbol{x}_t) - f_t(\boldsymbol{x}_{t+1}) \right) + R(\boldsymbol{x}) - R(\boldsymbol{x}_1),$$

where $\mathrm{regret}_T(\boldsymbol{x}) := \sum_{t=1}^{T}(f_t(\boldsymbol{x}_t) - f_t(\boldsymbol{x}))$.

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minimizer of $R(\cdot)$

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output of FTRL at $t + 1$

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- So our FTRL gives

$$\boldsymbol{x}_t = \arg\min_{\boldsymbol{x} \in \Delta} \left(\sum_{k=1}^{t-1} \langle \boldsymbol{\ell}_k, \boldsymbol{x} \rangle + c \cdot \sum_{i=1}^{n} \boldsymbol{x}(i) \ln \boldsymbol{x}(i) \right).$$

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- The constraint $\boldsymbol{x} \in \Delta \Rightarrow \sum_i \boldsymbol{x}_i = 1$.
- So we use Lagrange multiplier to solve

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- The partial derivative $\frac{\partial \mathcal{L}}{\partial \boldsymbol{x}(i)}$:

$$\left(\sum_{k=1}^{t-1} \boldsymbol{\ell}_k(i) \right) + c \cdot (1 + \ln \boldsymbol{x}_i) + \lambda$$

Rediscover MWU?

$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{x}(i)} = 0 \quad \Rightarrow \quad \boldsymbol{x}(i) = \exp\left(-1 - \frac{\lambda}{c} - \frac{1}{c}\sum_{k=1}^{t-1} \ell_k(i)\right)$$

Rediscover MWU?

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Exactly the solution of MWU if we take $c = 1/\beta$!

- Now it remains to bound the deviation of each step.

Regret of FTRL + Negative-Entropy Regularization

- At each step,
  $f_t(\boldsymbol{x}_t) - f_t(\boldsymbol{x}_{t+1}) = \langle \boldsymbol{\ell}_t, \boldsymbol{x}_t - \boldsymbol{x}_{t+1} \rangle$.
- Let's go back to use the notation of MWU.
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$\because$ weights are non-increasing

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assume $0 \leq \ell_t(i) \leq 1$

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Regret of FTRL + Negative-Entropy Regularization

- By Theorem 3, for any $\boldsymbol{x}$,

$$\text{regret}_T(\boldsymbol{x}) \le \sum_{t=1}^{T} (f_t(\boldsymbol{x}_t) - f_t(\boldsymbol{x}_{t+1})) + R(\boldsymbol{x}) - R(\boldsymbol{x}_1) \le \frac{T}{c} + c \ln n.$$

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$\because$ max entropy for uniform distribution

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Again, we have $\text{regret}_T \leq 2\sqrt{T \ln n}$ by choosing $c = \sqrt{\frac{T}{\ln n}}$.

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- Note the slight difference b/w regret and regret*.

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L2 Regularization

- Let's try to apply the FTRL to the case that the regularizer is of L2 norm!
- Consider also linear cost functions but $\mathcal{K} = \mathbb{R}^n$ first.
- What kind of problem we might encounter?

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- The offline optimum could be $-\infty$.
- FTL will also tend to find a solution of "big" size, too.
- To fight this tendency, it makes sense to use a regularizer which penalizes the size of a solution.

$$R(\boldsymbol{x}) := c||\boldsymbol{x}||^2.$$

The regularizer of 2-norm tells us...

- $\boldsymbol{x}_1 = \mathbf{0}$.
- $\boldsymbol{x}_{t+1} = \arg\min_{\boldsymbol{x}\in\mathbb{R}^n} c||\boldsymbol{x}||^2 + \sum_{k=1}^{t}\langle \boldsymbol{\ell}_k, \boldsymbol{x}\rangle$.

- Compute the gradient:

$$2c\boldsymbol{x} + \sum_{k=1}^{t} \boldsymbol{\ell}_k = 0$$

$$\Rightarrow \quad \boldsymbol{x} = -\frac{1}{2c}\sum_{k=1}^{t} \boldsymbol{\ell}_k.$$

Hence, $\boldsymbol{x}_1 = \mathbf{0}, \boldsymbol{x}_{t+1} = \boldsymbol{x}_t - \frac{1}{2c}\boldsymbol{\ell}_t$.

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$\rightarrow$ penalize the experts that performed badly in the past!

The regret of FTRL with 2-norm regularization

- First, we have

$$f_t(\boldsymbol{x}_t) - f_t(\boldsymbol{x}_{t+1}) = \langle \ell_t, \boldsymbol{x}_t - \boldsymbol{x}_{t+1} \rangle = \left\langle \ell_t, \frac{1}{2c}\ell_t \right\rangle = \frac{1}{2c}||\ell_t||^2.$$

- So, with respect to a solution $\boldsymbol{x}$,

$$\begin{aligned} \text{regret}_T(\boldsymbol{x}) &\leq R(\boldsymbol{x}) - R(\boldsymbol{x}_1) + \sum_{t=1}^{T} f_t(\boldsymbol{x}_t) - f_t(\boldsymbol{x}_{t+1}) \\ &= c||\boldsymbol{x}||^2 + \frac{1}{2c}\sum_{t=1}^{T}||\ell_t||^2. \end{aligned}$$

- Suppose that $||\ell_t|| \leq L$ for each t and $||\boldsymbol{x}|| \leq D$. Then by optimizing $c = \sqrt{\frac{T}{2D^2L^2}}$, we have

$$\text{regret}_T(\boldsymbol{x}) \leq DL\sqrt{2T}.$$

Dealing with constraints

- Let's deal with the constraint that $\mathcal{K}$ is an arbitrary convex set instead of $\mathbb{R}^n$.
- Using the same regularizer, we have our FTRL which gives

$$\boldsymbol{x}_1 = \arg\min_{\boldsymbol{x}\in\mathcal{K}} c||\boldsymbol{x}||^2,$$

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- **The idea:** First solve the unconstrained optimization and then project the solution on K.

Unconstrained optimization + projection

$$\boldsymbol{y}_{t+1} = \arg\min_{\boldsymbol{y}\in\mathbb{R}^n} c||\boldsymbol{y}||^2 + \sum_{k=1}^{t} \langle \ell_t, \boldsymbol{y} \rangle.$$

$$\boldsymbol{x}'_{t+1} = \Pi_{\mathcal{K}}(\boldsymbol{y}_{t+1}) = \arg\min_{\boldsymbol{x}\in\mathcal{K}} ||\boldsymbol{x} - \boldsymbol{y}_{t+1}||.$$

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- **Claim:** $\boldsymbol{x}'_{t+1} = \boldsymbol{x}_{t+1}$.

Proof of the claim: $\boldsymbol{x}'_{t+1} = \boldsymbol{x}_{t+1}$

- First, we already have that $\boldsymbol{y}_{t+1} = -\frac{1}{2c}\sum_{k=1}^{t} \ell_t$.
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To bound the regret

$$f_t(\boldsymbol{x}_t) - f_t(\boldsymbol{x}_{t+1}) = \langle \ell_t, \boldsymbol{x}_t - \boldsymbol{x}_{t+1} \rangle \quad \leq \quad ||\ell_t|| \cdot ||\boldsymbol{x}_t - \boldsymbol{x}_{t+1}||$$

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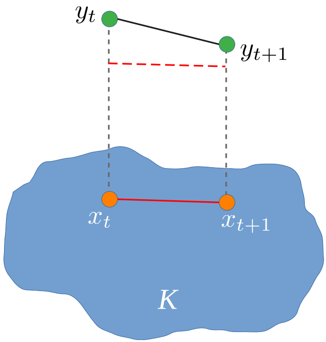

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Multi-Armed Bandit

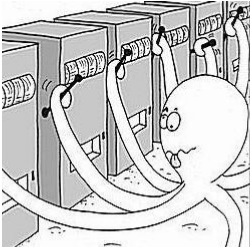

Fig.: Image credit: Microsoft Research

The setting

- We can see N arms as N experts.
- Arms give are independent.
- We can only pull an arm and observe the reward of it.
  - It's NOT possible to observe the reward of pulling the other arms...
- Each arm i has its own reward $r_i \in [0, 1]$.

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- Each arm i has its own reward $r_i \in [0, 1]$.
  - μ_i: the mean of reward of arm i
    - $\hat{\mu}_i$: the empirical mean of reward of arm i
  - μ^*: the mean of reward of the BEST arm.
  - $\Delta_i : \mu^* - \mu_i$.
  - Index of the best arm: $I^* := \arg\max_{i \in \{1,\ldots,N\}} \mu_i$.
  - The associated highest expected reward: $\mu^* = \mu_{I^*}$.

The regret formulation for MAB

Let I_t be the arm played by the algorithm at time t.
The regret of the algorithm in T rounds is

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A Naïve Greedy Algorithm

Greedy Algorithm

1. For $t \leq cN$, select a random arm with probability $1/N$ and pull it.
2. For $t > cN$, pull the arm $I_t := \arg\max_{i=1,\ldots,N} \hat{\mu}_{i,t}$.

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- Here c is a constant.
- This algorithm is of linear regret, hence is not a no-regret algorithm.
- For example,
  - Arm 1: $0/1$ reward with mean $3/4$.
  - Arm 2: Fixed reward of $1/4$.
  - After $cN = 2c$ steps, with constant probability, we have $\hat{\mu}_{1,cN} < \hat{\mu}_{2,cN}$.

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Greedy Algorithm

1. For $t \leq cN$, select a random arm with probability $1/N$ and pull it.
2. For $t > cN$, pull the arm $I_t := \arg\max_{i=1,\dots,N} \hat{\mu}_{i,t}$.

- Here c is a constant.
- This algorithm is of linear regret, hence is not a no-regret algorithm.
- For example,
  - Arm 1: $0/1$ reward with mean $3/4$.
  - Arm 2: Fixed reward of $1/4$.
  - After $cN = 2c$ steps, with constant probability, we have $\hat{\mu}_{1,cN} < \hat{\mu}_{2,cN}$.
  - If this is the case, the algorithm will keep pulling arm 2 and will never change!

ϵ-Greedy Algorithm

ϵ-Greedy Algorithm

For all $t = 1, 2, \ldots, N$:

- With probability $1 - \epsilon$, pull arm $I_t := \arg\max_{i=1,\ldots,N} \hat{\mu}_{i,t}$.
- With probability ϵ, select an arm uniformly at random (i.e., each with probability $1/N$).

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- Unfortunately, this algorithm still incurs linear regret.

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  - Each arm is pulled in average $\epsilon T/N$ times.

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- It looks good.
- Unfortunately, this algorithm still incurs linear regret.
- Indeed,
  - Each arm is pulled in average $\epsilon T/N$ times.
  - Hence the (expected) regret will be at least $\frac{\epsilon T}{N} \sum_{i:\mu_i < \mu^*} \Delta_i$.

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2. Gradient Descent for Online Convex Optimization (GD)
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The upper confidence bound algorithm (UCB)

- At each time step (round), we simply pull the arm with the highest "empirical reward estimate + high-confidence interval size".
- The empirical reward estimate of arm i at time t:

$$\hat{\mu}_{i,t} = \frac{\sum_{s=1}^{t} I_{s,i} \cdot r_s}{n_{i,t}}.$$

$n_{i,t}$: the number of times arm i is played.
$I_{s,i}$: 1 if the choice of arm is i at time s and 0 otherwise.

- Reward estimate + confidence interval:

$$\text{UCB}_{i,t} := \hat{\mu}_{i,t} + \sqrt{\frac{\ln t}{n_{i,t}}}.$$

Algorithm UCB

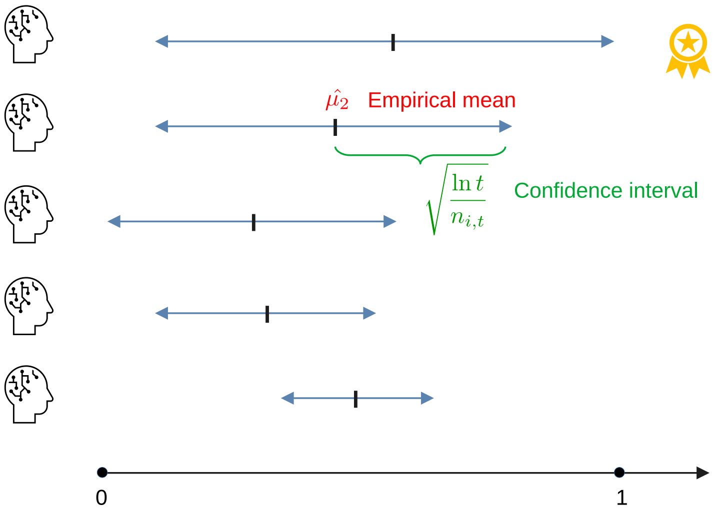
UCB Algorithm

N arms, T rounds such that $T \geq N$.

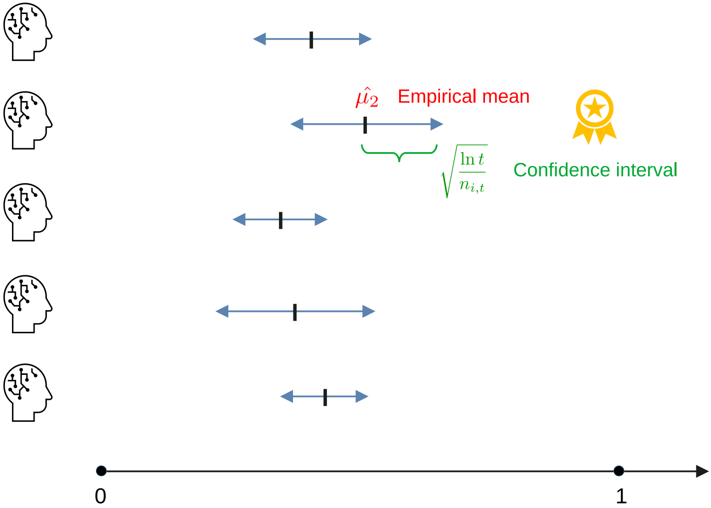
1. For $t = 1, \ldots, N$, play arm t.
2. For $t = N + 1, \ldots, T$, play arm

$$A_t = \arg\max_{i \in \{1,\ldots,N\}} \mathrm{UCB}_{i,t-1}.$$

Algorithm UCB



Algorithm UCB (after more time steps...)



From the Chernoff bound (proof skipped)

For each arm i at time t, we have

$$|\hat{\mu}_{i,t} - \mu_i| < \sqrt{\frac{\ln t}{n_{i,t}}}$$

with probability $\geq 1 - 2/t^2$.

Immediately, we know that

- with prob. $\geq 1 - 2/t^2$, $\mathrm{UCB}_{i,t} := \hat{\mu}_{i,t} + \sqrt{\frac{\ln t}{n_{i,t}}} > \mu_i$.
- with prob. $\geq 1 - 2/t^2$, $\hat{\mu}_{i,t} < \mu_i + \frac{\Delta_i}{2}$ when $n_{i,t} \geq \frac{4 \ln t}{\Delta_i^2}$.

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To understand why, please take my Randomized Algorithms course. :)
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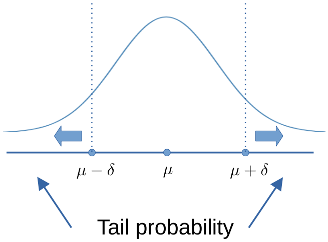
Appendix: Tail probability by the Chernoff/Hoeffding bound

The Chernoff/Hoeffding bound

For independent and identically distributed (i.i.d.) samples $x_1, \ldots, x_n \in [0, 1]$ with $\mathbb{E}[x_i] = \mu$, we have

$$\Pr\left[\left|\frac{\sum_{i=1}^{n} x_i}{n} - \mu\right| \geq \delta\right] \leq 2e^{-2n\delta^2}.$$



Very unlikely to play a suboptimal arm

Lemma 3

At any time step t, if a suboptimal arm i (i.e., $\mu_i < \mu^*$) has been played for $n_{i,t} \geq \frac{4 \ln t}{\Delta_i^2}$ times, then $\mathrm{UCB}_{i,t} < \mathrm{UCB}_{I^*,t}$ with probability $\geq 1 - 4/t^2$. Therefore, for any t,

$$\Pr\left[I_{t+1,i} = 1 \,\middle|\, n_{i,t} \geq \frac{4 \ln t}{\Delta_i^2} \right] \leq \frac{4}{t^2}.$$

Proof of Lemma 3

With probability $< 2/t^2 + 2/t^2$ (union bound) that

$$\begin{aligned}\mathrm{UCB}_{i,t} = \hat{\mu}_{i,t} + \sqrt{\frac{\ln t}{n_{i,t}}} &\leq \hat{\mu}_{i,t} + \frac{\Delta_i}{2} \\ &< \left(\mu_i + \frac{\Delta_i}{2}\right) + \frac{\Delta_i}{2} \\ &= \mu^* < \mathrm{UCB}_{i^*,t}\end{aligned}$$

does NOT hold.

Playing suboptimal arms for very limited number of times

Lemma 4

For any arm i with $\mu_i < \mu^*$,

$$\mathbb{E}[n_{i,T}] \leq \frac{4 \ln T}{\Delta_i^2} + 8.$$

$$\begin{aligned}
\mathbb{E}[n_{i,T}] &= 1 + \mathbb{E}\left[\sum_{t=N}^{T} \mathbb{1}\left\{I_{t+1,i} = 1\right\}\right] \\
&= 1 + \mathbb{E}\left[\sum_{t=N}^{T} \mathbb{1}\left\{I_{t+1,i} = 1, n_{i,t} < \frac{4 \ln t}{\Delta_i^2}\right\}\right] \\
&\quad + \mathbb{E}\left[\sum_{t=N}^{T} \mathbb{1}\left\{I_{t+1,i} = 1, n_{i,t} \geq \frac{4 \ln t}{\Delta_i^2}\right\}\right]
\end{aligned}$$

Proof of Lemma 4 (contd.)

$$
\begin{aligned}
\mathbb{E}[n_{i,T}] &\leq \frac{4\ln T}{\Delta_i^2} + \mathbb{E}\left[\sum_{t=N}^{T} \mathbb{1}\left\{I_{t+1,i} = 1, n_{i,t} \geq \frac{4\ln t}{\Delta_i^2}\right\}\right] \\
&= \frac{4\ln T}{\Delta_i^2} + \sum_{t=N}^{T} \Pr\left[I_{t+1,i} = 1, n_{i,t} \geq \frac{4\ln t}{\Delta_i^2}\right] \\
&= \frac{4\ln T}{\Delta_i^2} + \sum_{t=N}^{T} \Pr\left[I_{t+1,i} = 1 \,\middle|\, n_{i,t} \geq \frac{4\ln t}{\Delta_i^2}\right] \cdot \Pr\left[n_{i,t} \geq \frac{4\ln t}{\Delta_i^2}\right] \\
&\leq \frac{4\ln T}{\Delta_i^2} + \sum_{t=N}^{T} \frac{4}{t^2} \\
&\leq \frac{4\ln T}{\Delta_i^2} + 8.
\end{aligned}
$$

The regret bound for the UCB algorithm

Theorem 4

For all $T \geq N$, the (expected) regret by the UCB algorithm in round T is

$$\mathbb{E}[\text{regret}_T] \leq 5\sqrt{NT \ln T} + 8N.$$

Proof of Theorem 4

- Divide the arms into two groups:
  1. Group ONE (G_1): "almost optimal arms" with $\Delta_i < \sqrt{\frac{N}{T}\ln T}$.
  2. Group TWO (G_2): "bad" arms with $\Delta_i \geq \sqrt{\frac{N}{T}\ln T}$.

$$\sum_{i \in G_1} n_{i,T}\Delta_i \leq \left(\sqrt{\frac{N}{T}\ln T}\right)\sum_{i \in G_1} n_{i,T} \leq T \cdot \sqrt{\frac{N}{T}\ln T} = \sqrt{NT\ln T}.$$

By Lemma 4,

$$\begin{aligned}\sum_{i \in G_2} \mathbb{E}[n_{i,T}]\Delta_i \leq \sum_{i \in G_2} \frac{4\ln T}{\Delta_i} + 8\Delta_i &\leq \sum_{i \in G_2} 4\sqrt{\frac{T\ln T}{N}} + 8 \\ &\leq 4\sqrt{NT\ln T} + 8N.\end{aligned}$$

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Time Decaying ϵ-Greedy Algorithm

What if the horizon T is known in advance when we run ϵ-Greedy?

Time-Decaying ϵ-Greedy Algorithm

For all $t = 1, 2, \ldots, N$, set $\epsilon := N^{1/3}/T^{1/3}$:

- With probability $1 - \epsilon$, pull arm $I_t := \arg\max_{i=1,\ldots,N} \hat{\mu}_{i,t}$.
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- With probability ϵ, select an arm uniformly at random (i.e., each with probability $1/N$).

Claim

Time-Decaying ϵ-Greedy Algorithm gets roughly $O(N^{1/3} T^{2/3})$ regret.

Sketch of proving the claim

- The expected regret $\mathrm{E}[R(T)] = \sum_{t=1}^{T} \mathrm{E}[\mu^* - \mu_{T_t}]$.
- Using the greedy choice that $\hat{\mu}_{I_t} \geq \hat{\mu}_{I^*}$, we have

$$
\begin{aligned}
\mathrm{E}[R(T)] &\leq \sum_{t=1}^{T} (1-\epsilon)\mathrm{E}[(\mu_{I^*} - \hat{\mu}_{I^*} + \hat{\mu}_{I_t} - \mu_{I_t}) \mid \text{greedy choice of } I_t] + \epsilon T \\
&\leq \sum_{t=1}^{T} \left(\sqrt{\frac{\ln T}{n_{I^*,t}}} + \sqrt{\frac{\ln T}{n_{I_t,t}}} \right) + \frac{1}{T} \cdot 1 \cdot T + \epsilon T \quad \text{(Chernoff)} \\
&\approx\leq \sum_{t=1}^{T} \left(\sqrt{\frac{\ln T}{\epsilon t / N}} + \sqrt{\frac{\ln T}{\epsilon t / N}} \right) + \epsilon T + 1 \\
&\leq \sqrt{\frac{N}{\epsilon}} \sqrt{T \log T} + \epsilon T + 1 = O(N^{1/3} T^{2/3} \sqrt{\log T}).
\end{aligned}
$$

Thank you.