

A Sketch of Nash's Theorem from Fixed Point Theorems

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Reference

- ▶ Lecture Notes in 6.853 Topics in Algorithmic Game Theory [[link](#)].
- ▶ *Fixed Point Theorems and Applications to Game Theory*. Allen Yuan. The University of Chicago Mathematics REU 2017. [[link](#)].
 - ▶ REU = Research Experience for Undergraduate students.

Outline

Brouwer's Fixed Point Theorem

Sketch of the Proof of Nash's Theorem (from Brouwer's Theorem)

Kakutani's Fixed Point Theorem

Pure Nash Equilibria of Pure Strategic Games

Preliminaries

Main Theorem I & The Proof

Mixed Nash Equilibria of Finite Strategies Games

Preliminaries & Assumptions

Main Theorem II & the Proof

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The Setting

- ▶ A set N of n players.
- ▶ Strategy set $S_i = \{s_{i,1}, \dots, s_{i,k_i}\}$ for each player $i \in N$, k_i is bounded.
- ▶ Utility function: u_i for each player i .
- ▶ $\Delta := \Delta_1 \times \Delta_2 \times \dots \times \Delta_n$: a Cartesian product of $(\Delta_i)_{i \in N}$.
 - ▶ For $x \in \Delta$, $x_i(s)$ denotes the probability mass on strategy $s \in S_i$.
 - ▶ $\Delta_i = \{(x_i(s_{i,1}), x_i(s_{i,2}), \dots, x_i(s_{i,k_i})) \mid x_i(s_{i,j}) \geq 0 \forall j; \sum_j x_i(s_{i,j}) = 1\}$.
 - ▶ $x_i \in \Delta_i$: a **mixed strategy**.

Nash's Theorem

Nash (1950)

Every game $\langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$ has a Nash equilibrium.

► **Note:** $u_i(\mathbf{x}) := \sum_{s \in S_i} x_i(s) \cdot u_i(s; \mathbf{x}_{-i})$.

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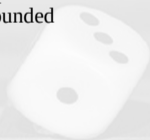
- ▶ **Note:** $u_i(\mathbf{x}) := \sum_{s \in S_i} x_i(s) \cdot u_i(s; \mathbf{x}_{-i})$.
- ▶ No player wants to deviate to the other strategy unilaterally.

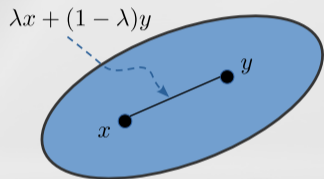


open &
bounded



closed &
bounded



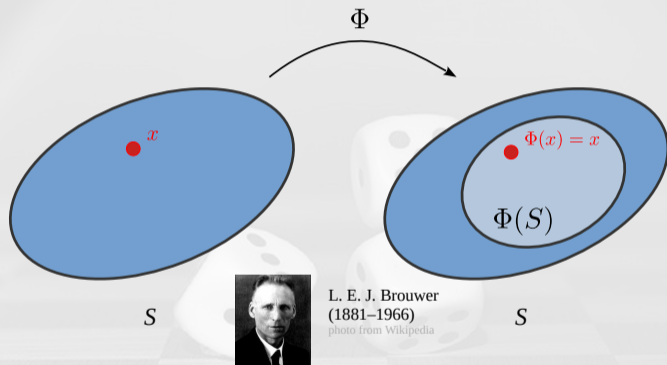


convex



not convex

Fixed Point



Brouwer's Fixed Point Theorem

Brouwer's Fixed-Point Theorem

Let D be a convex, compact (closed and bounded) subset of the Euclidean space. If $f : D \mapsto D$ is continuous, then there exists $x \in D$ such that

$$f(x) = x.$$

- ▶ **Idea:** We want the function f to satisfy the conditions of Brouwer's fixed point theorem.

Brouwer's Fixed Point Theorem

Brouwer's Fixed-Point Theorem

Let D be a convex, compact (closed and bounded) subset of the Euclidean space. If $f : D \mapsto D$ is continuous, then there exists $x \in D$ such that

$$f(x) = x.$$

- ▶ **Idea:** We want the function f to satisfy the conditions of Brouwer's fixed point theorem.
- ▶ Try to relate utilities of players to a function f like above.

The Gain function

Gain

Suppose that $\mathbf{x}' \in \Delta$ is given. For a player i and strategy $s_i \in S_i$ (or $s_i \in \Delta_i$), we define the **gain** as

$$\text{Gain}_{i,s_i}(\mathbf{x}') = \max\{u_i(s_i; \mathbf{x}'_{-i}) - u_i(\mathbf{x}), 0\},$$

which is non-negative.

- ▶ $\mathbf{x}'_{-i} := (x_j)_{j \in N}, (\mathbf{x}_{-i}, x_i) = \mathbf{x}$.
- ▶ It's equal to the increase in payoff for player i if he/she were to switch to pure strategy s_i .

Proof of Nash's Theorem (Define a response function)

- ▶ Define a function $f : \Delta \mapsto \Delta$ that satisfies the conditions of Brouwer's fixed point theorem.
- ▶ For all $\mathbf{x} \in \Delta$, $\mathbf{y} = f(\mathbf{x})$ where for all $i \in N$ and $s_i \in S_i$,

$$y_i(s_i) := \frac{x_i(s_i) + \text{Gain}_{i;s_i}(\mathbf{x})}{1 + \sum_{s'_i \in S_i} \text{Gain}_{i;s'_i}(\mathbf{x})}.$$

- ▶ f tries to boost the probability mass where strategy switching results in gains in payoff.

Proof of Nash's Theorem (Define a response function)

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- ▶ $f : \Delta \mapsto \Delta$ is continuous (verify this by yourself).
- ▶ Δ is a product of simplices so it is convex (verify this by yourself).
- ▶ Δ is closed and bounded, so it is compact.

Proof of Nash's Theorem (Define a response function)

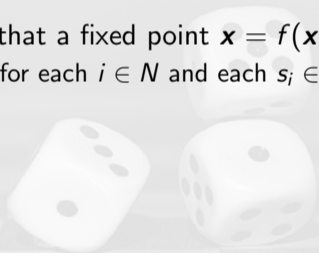
- ▶ Define a function $f : \Delta \mapsto \Delta$ that satisfies the conditions of Brouwer's fixed point theorem.
- ▶ For all $\mathbf{x} \in \Delta$, $\mathbf{y} = f(\mathbf{x})$ where for all $i \in N$ and $s_i \in S_i$,

$$y_i(s_i) := \frac{x_i(s_i) + \text{Gain}_{i;s_i}(\mathbf{x})}{1 + \sum_{s'_i \in S_i} \text{Gain}_{i;s'_i}(\mathbf{x})}.$$

- ▶ $f : \Delta \mapsto \Delta$ is continuous (verify this by yourself).
 - ▶ Δ is a product of simplices so it is convex (verify this by yourself).
 - ▶ Δ is closed and bounded, so it is compact.
- ★ Brouwer's fixed point theorem guarantees the existence of a fixed point of f .

Claim: Any fixed point of f is a Nash equilibrium

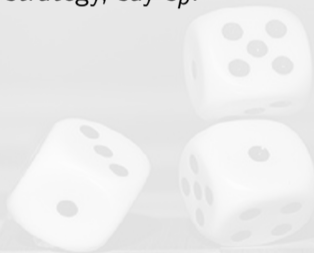
- ▶ It suffices to prove that a fixed point $\mathbf{x} = f(\mathbf{x})$ satisfies:
 - ▶ $\text{Gain}_{i;S_i}(\mathbf{x}) = 0$, for each $i \in N$ and each $s_i \in S_i$.



Claim: Any fixed point of f is a Nash equilibrium

Prove it by contradiction.

- ▶ Assume that there is some player p who can improve his/her payoff by switching to a pure strategy, say s_p :
 - ▶ $\text{Gain}_{p,s_p}(\mathbf{x}) > 0$.



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 - ▶ $\text{Gain}_{p;s_p}(\mathbf{x}) > 0$.
- ▶ Note that we must have $x_p(s_p) > 0$, otherwise \mathbf{x} cannot be a fixed point of f .
 - ▶ From the definition of f ; the numerator would be > 0 .

$$y_p(s_p) := \frac{x_p(s_p) + \text{Gain}_{p;s_p}(\mathbf{x})}{1 + \sum_{s'_p \in S_p} \text{Gain}_{p;s'_p}(\mathbf{x})}$$

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Prove it by contradiction.

- ▶ Assume that there is some player p who can improve his/her payoff by switching to a pure strategy, say s_p :
 - ▶ $\text{Gain}_{p;s_p}(\mathbf{x}) > 0 \Rightarrow u_p(s_p; \mathbf{x}_{-p}) - u_p(\mathbf{x}) > 0.$



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- ▶ $\text{Gain}_{p;s_p}(\mathbf{x}) > 0 \Rightarrow u_p(s_p; \mathbf{x}_{-p}) - u_p(\mathbf{x}) > 0.$

- ▶ We argue that there must be some other pure strategy \hat{s}_p such that:

- ▶ $x_p(\hat{s}_p) > 0$ and

- ▶ $u_p(\hat{s}_p; \mathbf{x}_{-p}) - u_p(\mathbf{x}) < 0$

★ Notice that

$$u_p(\mathbf{x}) := \sum_{s \in S_p} x_p(s) \cdot u_p(s; \mathbf{x}_{-p}).$$

Claim: Any fixed point of f is a Nash equilibrium

Prove it by contradiction.

- ▶ Assume that there is some player p who can improve his/her payoff by switching to a pure strategy, say s_p :

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- ▶ We argue that there must be some other pure strategy \hat{s}_p such that:

$$\text{▶ } x_p(\hat{s}_p) > 0 \text{ and}$$

$$\text{▶ } u_p(\hat{s}_p; \mathbf{x}_{-p}) - u_p(\mathbf{x}) < 0 \Rightarrow \text{Gain}_{p;\hat{s}_p}(\mathbf{x}) = 0.$$

★ Notice that

$$u_p(\mathbf{x}) := \sum_{s \in S_p} x_p(s) \cdot u_p(s; \mathbf{x}_{-p}).$$

- ▶ We obtain that (\mathbf{x} is not a fixed point $\Rightarrow \Leftarrow$)

$$y_p(\hat{s}_p) := \frac{x_p(\hat{s}_p) + \text{Gain}_{p;\hat{s}_p}(\mathbf{x})}{1 + \sum_{s'_p \in S_p} \text{Gain}_{p;s'_p}(\mathbf{x})} < x_p(\hat{s}_p).$$

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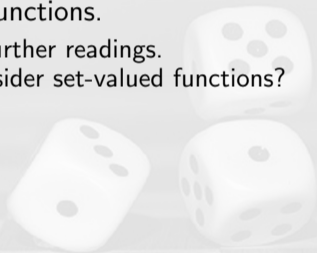
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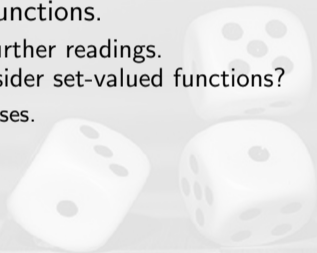
An Extension of Brouwer's work

- ▶ Focus: **set-valued** functions.
 - ▶ Refer here for further readings.
 - ▶ Why do we consider set-valued functions?



An Extension of Brouwer's work

- ▶ Focus: **set-valued** functions.
 - ▶ Refer here for further readings.
 - ▶ Why do we consider set-valued functions?
 - ▶ Best-responses.



Upper Semi-Continuous (having a closed graph)

Upper semi-continuous functions

Let

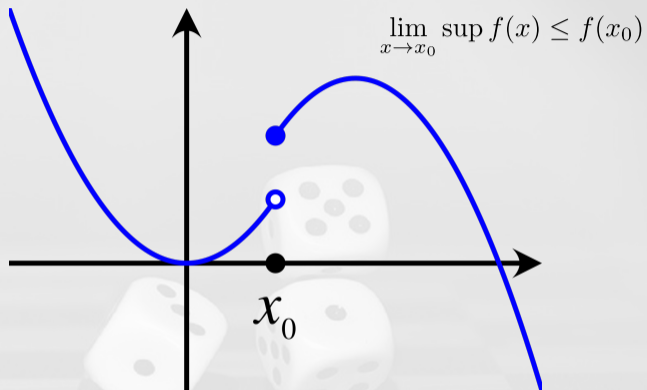
- ▶ $\mathbb{P}(X)$: all nonempty, closed, convex subsets of X .
- ▶ S : a nonempty, compact, and convex set.

Then the set-valued function $\Phi : S \mapsto \mathbb{P}(S)$ is **upper semi-continuous** if

for arbitrary sequences $(\mathbf{x}_n)_{n \in \mathbb{N}}, (\mathbf{y}_n)_{n \in \mathbb{N}}$ in S , we have

- ▶ $\lim_{n \rightarrow \infty} \mathbf{x}_n = \mathbf{x}_0$,
 - ▶ $\lim_{n \rightarrow \infty} \mathbf{y}_n = \mathbf{y}_0$,
 - ▶ $\mathbf{y}_n \in \Phi(\mathbf{x}_n)$ for all $n \in \mathbb{N}$,
- imply that $\mathbf{y}_0 \in \Phi(\mathbf{x}_0)$.

Removable discontinuity, Sequentially compact, Bolzano–Weierstrass theorem.



(Figure from Wikipedia)

Fixed Point of Set-Valued Functions

Fixed Point (Set-Valued)

A fixed point of a set-valued function $\Phi : S \mapsto \mathbb{P}(S)$ is a point $\mathbf{x}^* \in S$ such that $\mathbf{x}^* \in \Phi(\mathbf{x}^*)$.

Kakutani's Theorem for Simplices

Kakutani's Theorem for Simplices (1941)

If S is an r -dimensional closed simplex in a Euclidean space and $\Phi : S \mapsto \mathbb{P}(S)$ is upper semi-continuous, then Φ has a fixed point.

Kakutani's Fixed-Point Theorem

Kakutani's Fixed-Point Theorem (1941)

If S is a **nonempty, compact, convex set** in a Euclidean space and $\Phi : S \mapsto \mathbb{P}(S)$ is upper semi-continuous, then Φ has a fixed point.



Kakutani's Fixed-Point Theorem

Kakutani's Fixed-Point Theorem (1941)

If S is a **nonempty, compact, convex set** in a Euclidean space and $\Phi : S \mapsto \mathbb{P}(S)$ is upper semi-continuous, then Φ has a fixed point.

- ▶ We won't go over its proof.
- ▶ Instead, we will delve into how it can be used to prove Nash's Theorem from the perspectives of set-valued functions and best-responses.

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Cartesian product of Sets

Cartesian Product

For a family of sets $\{A_i\}_{i \in N}$, $\prod_{i \in N} A_i = A_1 \times A_2 \times \cdots \times A_n$ denotes the Cartesian product of A_i for $i \in N$.

Profile

for $x_i \in A_i$, then $(x_i)_{i \in N}$ is called a (strategy) profile.

Binary Relation

Binary Relation

- ▶ A binary relation on a set A is a subset of $A \times A$ consisting of all pairs of elements.
- ▶ For $a, b \in A$, we denote by $R(a, b)$ if a is related to b .



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Properties on Binary Relations

- ▶ **Completeness:** For all $a, b \in A$, we have $R(a, b)$, $R(b, a)$, or both.
- ▶ **Reflexivity:** For all $a \in A$, we have $R(a, a)$.
- ▶ **Transitivity:** For $a, b, c \in A$, if $R(a, b)$ and $R(b, c)$, then we have $R(a, c)$.

Preference Relation

Preference Relation

A preference relation is a **complete, reflexive, and transitive** binary relation.

- ▶ Denote by $a \succsim b$ if a is related to b .
- ▶ Denote by $a \succ b$ if $a \succsim b$ but $b \not\succsim a$.
- ▶ Denote by $a \sim b$ if $a \succsim b$ and $b \succsim a$.

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-
- ▶ $a \succsim b$: a is **weakly preferred to** b .
 - ▶ $a \sim b$: agent is indifferent between a and b .

Continuity on a Preference relation

Continuous Preference Relation

A preference relation is **continuous** if:

whenever there exist sequences $(a_k)_{k \in \mathbb{N}}$ and $(b_k)_{k \in \mathbb{N}}$ in A such that

- ▶ $\lim_{k \rightarrow \infty} a_k = a$,
- ▶ $\lim_{k \rightarrow \infty} b_k = b$,
- ▶ and $a_k \succsim b_k$ for all $k \in \mathbb{N}$

we have $a \succsim b$.

Strategic Games

Strategic Games

A strategic game is a tuple $\langle N, (A_i), (\succsim_i) \rangle$ consisting of

- ▶ a finite set of **players** N .
 - ▶ for each player $i \in N$, a nonempty set of **actions** A_i .
 - ▶ for each player $i \in N$, a **preference relation** \succsim_i on $A = \prod_{j \in N} A_j$.
- ▶ A strategic game is **finite** if A_i is finite for all $i \in N$.

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-
- ▶ A strategic game is **finite** if A_i is finite for all $i \in N$.
 - ▶ **Note:** \succsim_i is not defined on A_i only, but instead on the set of all $(A_j)_{j \in N}$.

PNE w.r.t. a Preference Relation

Pure Nash Equilibrium (PNE) with (\succsim_i)

A (pure) Nash equilibrium (PNE) of a strategic game $\langle N, (A_i), (\succsim_i) \rangle$ is a profile $\mathbf{a}^* := (a_i)_{i \in N}$ such that for all $i \in N$, we have

$$(\mathbf{a}_{-i}^*, a_i^*) \succsim_i (\mathbf{a}_{-i}^*, a'_i) \text{ for all } a'_i \in A_i.$$

Best-Response Function

Best-Response Functions

The **best-response** function of player i ,

$$BR_i : \prod_{j \in N \setminus \{i\}} A_j \mapsto \mathbb{P}(A_i),$$

is given by

$$BR_i(\mathbf{a}_{-i}) = \{a_i \in A_i \mid (\mathbf{a}_{-i}, a_i) \succeq_i (\mathbf{a}_{-i}, a'_i) \text{ for all } a'_i \in A_i\}.$$

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- ▶ BR_i is set-valued.
- ▶ Recall: $\mathbb{P}(X)$ includes all nonempty, closed, and convex subsets of X .

PNE w.r.t. a Preference Relation

- ▶ Alternative definition of NE.

Pure Nash Equilibrium (PNE) with (\succsim_i)

A Nash equilibrium of a strategic game $\langle N, (A_i), (\succsim_i) \rangle$ is a profile $\mathbf{a}^* := (a_i)_{i \in N}$ such that $a_i^* \in BR_i(\mathbf{a}_{-i}^*)$ for all $i \in N$.

- ▶ Thus, to prove the existence of a PNE for a strategic game $\langle N, (A_i), (\succsim_i) \rangle$, it suffices to show that:

PNE w.r.t. a Preference Relation

- ▶ Alternative definition of NE.

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- ▶ Thus, to prove the existence of a PNE for a strategic game $\langle N, (A_i), (\succsim_i) \rangle$, it suffices to show that:
 - ▶ There exists a profile $\mathbf{a}^* \in A$ such that for all $i \in N$ we have $a_i^* \in BR_i(\mathbf{a}_{-i}^*)$.

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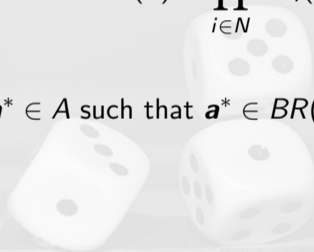
Main Theorem II & the Proof

General Idea

- ▶ Let $BR : A \mapsto \mathbb{P}(A)$ be

$$BR(\mathbf{a}) = \prod_{i \in N} BR_i(\mathbf{a}_{-i}).$$

- ▶ We seek for some $\mathbf{a}^* \in A$ such that $\mathbf{a}^* \in BR(\mathbf{a}^*)$.



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- ▶ We can then use Kakutani's Fixed-Point Theorem to show that \mathbf{a}^* exists.

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- ▶ We seek for some $\mathbf{a}^* \in A$ such that $\mathbf{a}^* \in BR(\mathbf{a}^*)$.
- ▶ We can then use Kakutani's Fixed-Point Theorem to show that \mathbf{a}^* exists.
- ▶ Yet, we need to verify the conditions under which Kakutani's Fixed-Point Theorem holds.

Quasi-Concave

Quasi-Concave of \succsim_i

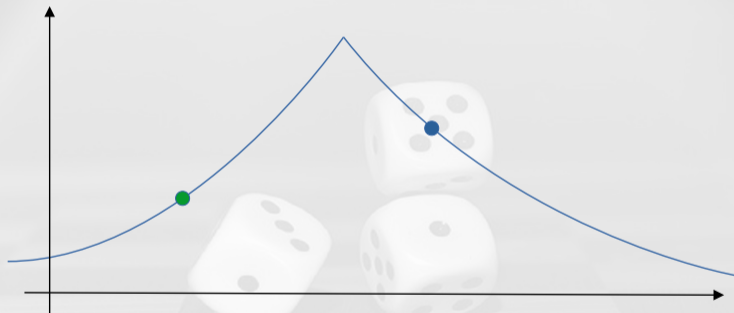
A preference relation \succsim_i over A is **quasi-concave** on A_i if for all $\mathbf{a} \in A$, the set

$$\{\mathbf{a}'_i \in A_i \mid (\mathbf{a}_{-i}, \mathbf{a}'_i) \succsim_i (\mathbf{a}_{-i}, \mathbf{a}_i)\}$$

is **convex**.

- ▶ Then, we can consider the following theorem which guarantees the condition of a PNE.

An example of quasi-concave function.



$$f(\lambda x + (1 - \lambda)y) \geq \min\{f(x), f(y)\}, \text{ for } \lambda \in [0, 1]$$

The Main Theorem I

Main Theorem I

The strategic game $\langle N, (A_i), (\succsim_i) \rangle$ has a (pure) Nash equilibrium if

- ▶ A_i is a nonempty, compact, and convex subset of a Euclidean space
- ▶ \succsim_i is continuous and quasi-concave on A_i for all $i \in N$.
- ▶ We will show that A (cf. S) and BR (cf. Φ) satisfy the conditions to apply Kakutani's Fixed-Point Theorem.

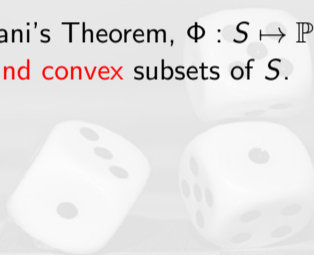
Requirements for A & BR

- ▶ A_i is nonempty, compact and convex for all $i \in N$, so their Cartesian product (i.e., A) must also be nonempty, compact and convex.



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- ▶ Note that in Kakutani's Theorem, $\Phi : S \mapsto \mathbb{P}(S)$, where $\mathbb{P}(S)$ denotes all **nonempty, closed, and convex** subsets of S .



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- ▶ We need to show that $BR_i(\mathbf{a}_{-i})$ is nonempty, closed, and convex for all $\mathbf{a}_{-i} \in \prod_{j \in N \setminus \{i\}} A_j$.

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- ▶ We need to show that $BR_i(\mathbf{a}_{-i})$ is nonempty, closed, and convex for all $\mathbf{a}_{-i} \in \prod_{j \in N \setminus \{i\}} A_j$.
 - ▶ Their Cartesian product $BR(\mathbf{a})$ is then nonempty, closed and convex, too.
 - ▶ We then have $BR : A \mapsto \mathbb{P}(A)$.

$BR_i(\mathbf{a}_{-i})$ is nonempty

- ▶ Assume that we can construct a continuous function (utility function) $u_i : A_i \mapsto \mathbb{R}$ such that for $a_i, a'_i \in A_i$, $(\mathbf{a}_{-i}, a_i) \succsim (\mathbf{a}_{-i}, a'_i)$ if and only if $u_i(a_i) \geq u_i(a'_i)$.



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- ▶ Since A_i is compact and u_i is continuous, $u_i(A_i)$ is compact as well.
- ▶ By the **Extreme Value Theorem**, there must exist some $a_i^* \in A_i$ such that $u_i(a_i^*) \geq u_i(a_i)$ for all $a_i \in A_i$.

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- ▶ By definition of u_i , it follows that $(\mathbf{a}_{-i}, a_i^*) \succsim (\mathbf{a}_{-i}, a_i)$ for all $a_i \in A_i$, thus $a_i^* \in BR_i(\mathbf{a}_{-i})$.

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- ▶ So $BR_i(\mathbf{a}_{-i})$ is nonempty.

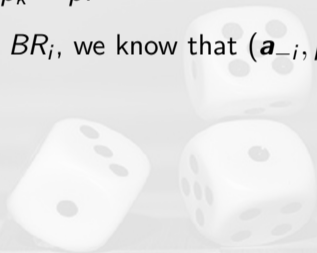
$BR_i(\mathbf{a}_{-i})$ is closed

- ▶ Take an arbitrary $p \in \overline{BR_i(\mathbf{a}_{-i})}$.
- ▶ There must exist some sequence $(p_k)_{k \in \mathbb{N}}$ such that $p_k \in BR_i(\mathbf{a}_{-i})$ for all $k \in \mathbb{N}$ and $\lim_{k \rightarrow \infty} p_k = p$.



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 - ▶ By the continuity of \succsim_i , we have $(\mathbf{a}_{-i}, p) \succsim_i (\mathbf{a}_{-i}, a_i)$ for all $a_i \in A_i$.

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- ▶ By the definition of BR_i , we know that $(\mathbf{a}_{-i}, p_k) \succsim_i (\mathbf{a}_{-i}, a_i)$ for all $a_i \in A_i$.
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 - ▶ By the continuity of \succsim_i , we have $(\mathbf{a}_{-i}, p) \succsim_i (\mathbf{a}_{-i}, a_i)$ for all $a_i \in A_i$.
 - $\Rightarrow p \in BR_i(\mathbf{a}_{-i})$ ($\because BR_i(\mathbf{a}_{-i})$ is closed).

$BR_i(\mathbf{a}_{-i})$ is convex

- ▶ Consider $a_i \in BR_i(\mathbf{a}_{-i})$.
- ▶ π_i is quasi-concave on $A_i \Rightarrow$



$BR_i(\mathbf{a}_{-i})$ is convex

- ▶ Consider $a_i \in BR_i(\mathbf{a}_{-i})$.
- ▶ \succsim_i is quasi-concave on $A_i \Rightarrow$

$$S = \{a'_i \in A_i \mid (\mathbf{a}_{-i}, a'_i) \succsim_i (\mathbf{a}_{-i}, a_i)\} \text{ is convex}$$

- ▶ Since a_i is a best response, the responses a'_i weakly preferable to a_i must be also best responses.

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- ▶ Any other best response $a_i^* \in BR_i(\mathbf{a}_{-i})$ must be at least good as a_i

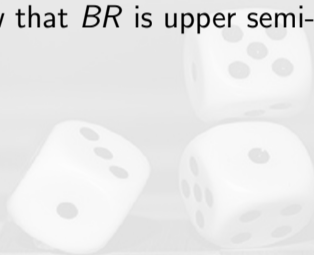
$BR_i(\mathbf{a}_{-i})$ is convex

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- ▶ Any other best response $a_i^* \in BR_i(\mathbf{a}_{-i})$ must be at least good as $a_i \Rightarrow BR_i(\mathbf{a}_{-i}) \subseteq S$.
- ▶ Hence, we have $BR_i(\mathbf{a}_{-i}) = S$, so $BR_i(\mathbf{a}_{-i})$ is convex.

- ▶ Next, we will show that BR is upper semi-continuous.



Recall: Upper Semi-Continuous

Upper semi-continuous functions

Let

- ▶ $\mathbb{P}(X)$: all nonempty, closed, convex subsets of X .
- ▶ S : a nonempty, compact, and convex set.

Then the set-valued function $\Phi : S \mapsto \mathbb{P}(S)$ is **upper semi-continuous** if

for arbitrary sequences $(\mathbf{x}_n)_{n \in \mathbb{N}}, (\mathbf{y}_n)_{n \in \mathbb{N}}$ in S , we have

- ▶ $\lim_{n \rightarrow \infty} \mathbf{x}_n = \mathbf{x}_0$,
 - ▶ $\lim_{n \rightarrow \infty} \mathbf{y}_n = \mathbf{y}_0$,
 - ▶ $\mathbf{y}_n \in \Phi(\mathbf{x}_n)$ for all $n \in \mathbb{N}$,
- imply that $\mathbf{y}_0 \in \Phi(\mathbf{x}_0)$.

BR is upper semi-continuous

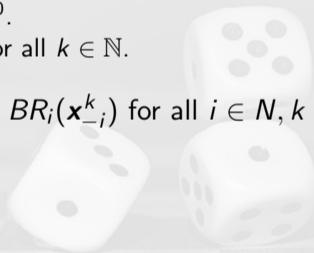
- ▶ Consider two sequences $(\mathbf{x}^k), (\mathbf{y}^k)$ in A such that

$$\lim_{k \rightarrow \infty} \mathbf{x}^k = \mathbf{x}^0,$$

$$\lim_{k \rightarrow \infty} \mathbf{y}^k = \mathbf{y}^0.$$

$$\mathbf{y}^k \in BR_i(\mathbf{x}^k) \text{ for all } k \in \mathbb{N}.$$

- ▶ Then we have $y_i^k \in BR_i(\mathbf{x}_{-i}^k)$ for all $i \in N, k \in \mathbb{N}$.



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$$\mathbf{y}^k \in BR_i(\mathbf{x}^k) \text{ for all } k \in \mathbb{N}.$$

- ▶ Then we have $y_i^k \in BR_i(\mathbf{x}_{-i}^k)$ for all $i \in N, k \in \mathbb{N}$.
- ▶ For an arbitrary $i \in N$, we have $(\mathbf{x}_{-i}^k, y_i^k) \succeq_i (\mathbf{x}_{-i}^k, a_i)$ for all $a_i \in A_i$ and $k \in \mathbb{N}$ (\cdot : best response).

BR is upper semi-continuous (contd.)

- ▶ For each $a_i \in A_i$, we can construct:
 - ▶ a sequence $((\mathbf{x}_{-i}^k, y_i^k))_{k \in \mathbb{N}}$ such that $\lim_{k \rightarrow \infty} (\mathbf{x}_{-i}^k, y_i^k) = (\mathbf{x}_{-i}^0, y_i^0)$.
 - ▶ a sequence $((\mathbf{x}_{-i}^k, a_i))_{k \in \mathbb{N}}$ such that $\lim_{k \rightarrow \infty} (\mathbf{x}_{-i}^k, a_i) = (\mathbf{x}_{-i}^0, a_i)$.
- ▶ Note that we have $(\mathbf{x}_{-i}^k, y_i^k) \succsim_i (\mathbf{x}_{-i}^k, a_i)$ for all $k \in \mathbb{N}$.
 - ▶ By continuity of \succsim_i , we have $(\mathbf{x}_{-i}^0, y_i^0) \succsim_i (\mathbf{x}_{-i}^0, a_i)$ for all $a_i \in A_i$.



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 - ▶ By continuity of \succeq_i , we have $(\mathbf{x}_{-i}^0, y_i^0) \succeq_i (\mathbf{x}_{-i}^0, a_i)$ for all $a_i \in A_i$.
- ▶ Thus, we have $y_i^0 \in BR_i(\mathbf{x}_{-i}^0)$ for all $i \in N$.
 - ▶ $\mathbf{y}^0 \in BR(\mathbf{x}^0)$.

BR is upper semi-continuous (contd.)

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- ▶ Thus, we have $y_i^0 \in BR_i(\mathbf{x}_{-i}^0)$ for all $i \in N$.
 - ▶ $\mathbf{y}^0 \in BR(\mathbf{x}^0)$.
- ▶ Therefore, BR is upper semi-continuous.

BR is upper semi-continuous (contd.)

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- ▶ Thus, we have $y_i^0 \in BR_i(\mathbf{x}_{-i}^0)$ for all $i \in N$.
 - ▶ $\mathbf{y}^0 \in BR(\mathbf{x}^0)$.
- ▶ Therefore, BR is upper semi-continuous.

By Kakutani's Fixed-Point Theorem, there exists some $\mathbf{a}^* \in A$ such that $\mathbf{a}^* \in BR(\mathbf{a}^*)$

BR is upper semi-continuous (contd.)

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 - ▶ a sequence $((\mathbf{x}_{-i}^k, a_i))_{k \in \mathbb{N}}$ such that $\lim_{k \rightarrow \infty} (\mathbf{x}_{-i}^k, a_i) = (\mathbf{x}_{-i}^0, a_i)$.
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- ▶ Thus, we have $y_i^0 \in BR_i(\mathbf{x}_{-i}^0)$ for all $i \in N$.
 - ▶ $\mathbf{y}^0 \in BR(\mathbf{x}^0)$.
- ▶ Therefore, BR is upper semi-continuous.

By Kakutani's Fixed-Point Theorem, there exists some $\mathbf{a}^* \in A$ such that $\mathbf{a}^* \in BR(\mathbf{a}^*) \Rightarrow \mathbf{a}^*$ is a PNE of the strategic game.

Outline

Brouwer's Fixed Point Theorem

Sketch of the Proof of Nash's Theorem (from Brouwer's Theorem)

Kakutani's Fixed Point Theorem

Pure Nash Equilibria of Pure Strategic Games

Preliminaries

Main Theorem I & The Proof

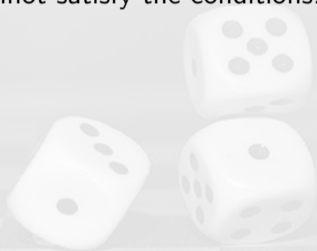
Mixed Nash Equilibria of Finite Strategies Games

Preliminaries & Assumptions

Main Theorem II & the Proof

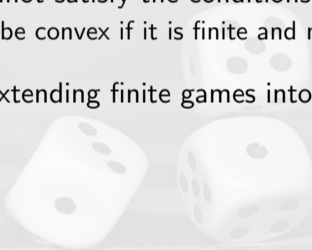
Limitations of the Previous PNE Result

- ▶ Any **finite** game cannot satisfy the conditions.



Limitations of the Previous PNE Result

- ▶ Any **finite** game cannot satisfy the conditions.
 - ▶ Each A_i cannot be convex if it is finite and nonempty.
- ★ Next, we consider extending finite games into **non-deterministic (randomized)** strategies.



Assumptions

- ▶ For a strategic game $\langle N, (A_i), (\succsim_i) \rangle$, we assume that we can construct a utility function $u_i : A \mapsto \mathbb{R}$, where $A = \prod_{i \in N} A_i$.
- ▶ Each player's *expected utility* is coupled with the set of probability distributions over A .
- ▶ $\Delta(X)$: the set of probability distributions over X .
- ▶ If X is finite and $\delta \in \Delta(X)$, then
 - ▶ $\delta(x)$: the probability that δ assigns to $x \in X$.
 - ▶ The support of δ : $\chi(\delta) = \{x \in X \mid \delta(x) > 0\}$.

Mixed Strategy

Mixed Strategy

Given a strategic game $\langle N, (A_i), (u_i) \rangle$, we call

- ▶ $\alpha_i \in \Delta(A_i)$ a **mixed strategy**.
- ▶ $a_i \in A_i$ a **pure strategy**.

A profile of mixed strategies $\alpha = (\alpha_j)_{j \in N}$ induces a probability distribution over A .

- ▶ The probability of $\mathbf{a} = (a_j)_{j \in N}$ under α :

$$\alpha(\mathbf{a}) = \prod_{j \in N} \alpha_j(a_j). \quad (\text{a normal product})$$

(A_i is finite $\forall i \in N$ and each player's strategy is resolved independently.)

prob. = $\alpha_1(t_1) \cdot \alpha_2(s_1)$

$\alpha_2(s_1)$

$\alpha_2(s_2)$

s_1

s_2

$\alpha_1(t_1)$ t_1

$u_1(t_1, s_1), u_2(t_1, s_1)$

$u_1(t_1, s_2), u_2(t_1, s_2)$

$\alpha_1(t_2)$ t_2

$u_1(t_2, s_1), u_2(t_2, s_1)$

$u_1(t_2, s_2), u_2(t_2, s_2)$

Mixed Extension of $\langle N, (A_i), (u_i) \rangle$

Mixed Extension of the Strategic Games

$\langle N, (\Delta(A_i)), (U_i) \rangle$:

- ▶ $U_i : \prod_{i \in N} \Delta(A_i) \mapsto \mathbb{R}$; expected utility over A induced by $\alpha \in \prod_{i \in N} \Delta(A_i)$.
- ▶ If A_j is finite for all $j \in N$, then

$$\begin{aligned} U_i(\alpha) &= \sum_{\mathbf{a} \in A} (\alpha(\mathbf{a}) \cdot u_i(\mathbf{a})) \\ &= \sum_{\mathbf{a} \in A} \left(\left(\prod_{j \in N} \alpha_j(a_j) \right) \cdot u_i(\mathbf{a}) \right). \end{aligned}$$

Main Theorem II

Main Theorem II

Every finite strategies game has a mixed strategy Nash equilibrium.

- ▶ Consider an arbitrary finite strategic game $\langle N, (A_i), (u_i) \rangle$, let $m_i := |A_i|$ for all $i \in N$.
- ▶ Represent each $\Delta(A_i)$ as a collection of vectors $\mathbf{p}^i = (p_1, p_2, \dots, p_{m_i})$.
 - ▶ $p_k \geq 0$ for all $k \in [m_i]$ and $\sum_{k=1}^{m_i} p_k = 1$.
 - ▶ $\Delta(A_i)$ is a standard $m_i - 1$ simplex for all $i \in N$.

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 - ▶ $p_k \geq 0$ for all $k \in [m_i]$ and $\sum_{k=1}^{m_i} p_k = 1$.
 - ▶ $\Delta(A_i)$ is a standard $m_i - 1$ simplex for all $i \in N$.
 - ★ $\Delta(A_i)$: nonempty, compact, and convex for each $i \in N$.
- ▶ U_i : continuous (\cdot : multilinear).
- ▶ Next, we show that U_i is **quasi-concave** in $\Delta(A_i)$.

Proof of Main Theorem II (contd.)

- ▶ Consider $\alpha \in \prod_{i \in N} \Delta(A_i)$.
- ▶ **Goal:** Show that $S = \{\alpha'_i \in \Delta(A_i) \mid U_i(\alpha_{-i}, \alpha'_i) \geq U_i(\alpha_{-i}, \alpha_i)\}$ is convex.



Proof of Main Theorem II (contd.)

- ▶ Consider $\alpha \in \prod_{i \in N} \Delta(A_i)$.
- ▶ **Goal:** Show that $S = \{\alpha'_i \in \Delta(A_i) \mid U_i(\alpha_{-i}, \alpha'_i) \geq U_i(\alpha_{-i}, \alpha_i)\}$ is convex.
- ▶ Take $\beta_i, \gamma_i \in S$, $\lambda \in [0, 1]$.
- ▶ By definition of S , we have
 - ▶ $U_i(\alpha_{-i}, \beta_i) \geq U_i(\alpha_{-i}, \alpha_i)$, and
 - ▶ $U_i(\alpha_{-i}, \gamma_i) \geq U_i(\alpha_{-i}, \alpha_i)$.

Proof of Main Theorem II (contd.)

- ▶ Consider $\alpha \in \prod_{i \in N} \Delta(A_i)$.
- ▶ **Goal:** Show that $S = \{\alpha'_i \in \Delta(A_i) \mid U_i(\alpha_{-i}, \alpha'_i) \geq U_i(\alpha_{-i}, \alpha_i)\}$ is convex.
- ▶ Take $\beta_i, \gamma_i \in S$, $\lambda \in [0, 1]$.
- ▶ By definition of S , we have
 - ▶ $U_i(\alpha_{-i}, \beta_i) \geq U_i(\alpha_{-i}, \alpha_i)$, and
 - ▶ $U_i(\alpha_{-i}, \gamma_i) \geq U_i(\alpha_{-i}, \alpha_i)$.
- ▶ $\lambda U_i(\alpha_{-i}, \beta_i) + (1 - \lambda) U_i(\alpha_{-i}, \gamma_i) \geq \lambda U_i(\alpha_{-i}, \alpha_i) + (1 - \lambda) U_i(\alpha_{-i}, \alpha_i) = U_i(\alpha_{-i}, \alpha_i)$.

Proof of Main Theorem II (contd.)

- ▶ By the multilinearity of U_i , we have

$$\lambda U_i(\alpha_{-i}, \beta_i) + (1 - \lambda) U_i(\alpha_{-i}, \gamma_i) = U_i(\alpha_{-i}, \lambda \beta_i + (1 - \lambda) \gamma_i).$$



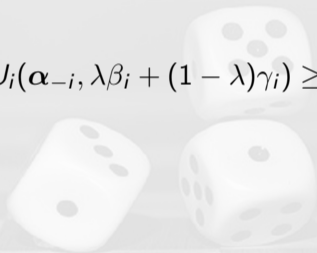
Proof of Main Theorem II (contd.)

- ▶ By the multilinearity of U_i , we have

$$\lambda U_i(\alpha_{-i}, \beta_i) + (1 - \lambda) U_i(\alpha_{-i}, \gamma_i) = U_i(\alpha_{-i}, \lambda \beta_i + (1 - \lambda) \gamma_i).$$

- ▶ So,

$$U_i(\alpha_{-i}, \lambda \beta_i + (1 - \lambda) \gamma_i) \geq U_i(\alpha_{-i}, \alpha_i).$$



Proof of Main Theorem II (contd.)

- ▶ By the multilinearity of U_i , we have

$$\lambda U_i(\alpha_{-i}, \beta_i) + (1 - \lambda) U_i(\alpha_{-i}, \gamma_i) = U_i(\alpha_{-i}, \lambda \beta_i + (1 - \lambda) \gamma_i).$$

- ▶ So,

$$U_i(\alpha_{-i}, \lambda \beta_i + (1 - \lambda) \gamma_i) \geq U_i(\alpha_{-i}, \alpha_i).$$

$$\lambda \beta_i + (1 - \lambda) \gamma_i \in S$$

Proof of Main Theorem II (contd.)

- ▶ By the multilinearity of U_i , we have

$$\lambda U_i(\alpha_{-i}, \beta_i) + (1 - \lambda) U_i(\alpha_{-i}, \gamma_i) = U_i(\alpha_{-i}, \lambda\beta_i + (1 - \lambda)\gamma_i).$$

- ▶ So,

$$U_i(\alpha_{-i}, \lambda\beta_i + (1 - \lambda)\gamma_i) \geq U_i(\alpha_{-i}, \alpha_i).$$

$$\lambda\beta_i + (1 - \lambda)\gamma_i \in S \Rightarrow U_i \text{ is convex.}$$

- ▶ Thus, U_i is quasi-concave in $\Delta(A_i)$.

We are done.

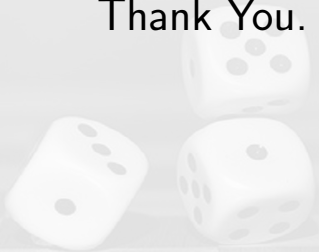
A Question

Matching Pennies of Infinite Actions

We have two players A and B having utility functions $f(x, y) = (x - y)^2$ and $g(x, y) = -(x - y)^2$ respectively. $x, y \in [-1, 1]$.

- ▶ Does this game has a pure Nash equilibrium?
- ▶ Why can't we use Kakutani's fixed point theorem?

Thank You.

Three white dice are scattered on a checkered board. One die is in the foreground, showing a one and a two. Another die is behind it, showing a one and a six. A third die is to the right, showing a one and a five. The background is a light gray gradient.