The Graph Abstract Data Type

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Outline

1. Introduction
   - Motivating Examples
   - Graphs
2. Graph Representations

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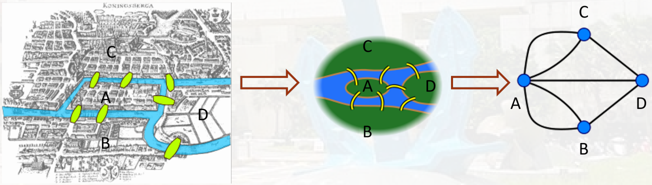
Königsberg Bridge Problem

Question

Can we walk across all the bridges exactly once in returning back to the starting land area?



- Recall **graphs** from the Discrete Mathematics course.
  - Land $\mapsto$ vertex
  - Bridge $\mapsto$ edge

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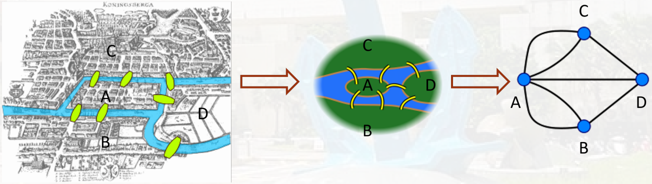

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Eulerian Walk

Euler's Theorem

A connected graph has an Euler cycle if and only if every vertex has even degree.

- **degree** of vertex v: number of neighbors of v in the graph.
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So, what about the answer to the Königberg Bridge Problem?

Definition of a Graph

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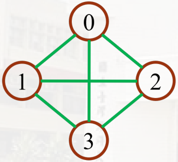
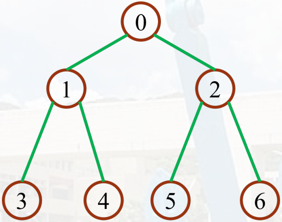
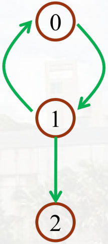
Examples



G_1



G_2



G_3

$V(G_1)=\{0, 1, 2, 3\}$
$E(G_1)=\{(0, 1), (0, 2), (0, 3), (1, 2), (1, 3), (2, 3)\}$

$V(G_2)=\{0, 1, 2, 3, 4, 5, 6\}$
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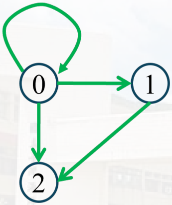
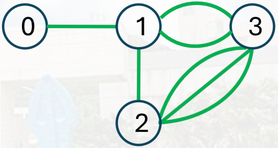
$V(G_3)=\{0, 1, 2\}$
$E(G_3)=\{<0, 1>, <1, 0>, <1, 2>\}$

Self-Loop & Multigrpah



Graph with a self-loop



multigraph

- (v, v): self-loop.
- multigraph: a graph with multiple occurrence of some edges.

Complete Graph

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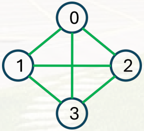
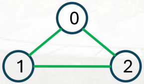
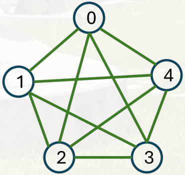






Subgraph and Induced Subgraph

- If (u, v) is an edge in $E(G)$, then the vertices u and v are adjacent and that the edge (u, v) is incident on vertices u and v.

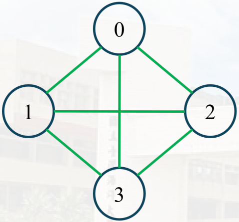
Subgraph

A subgraph of G is a graph G' such that $V(G') \subseteq V(G)$ and $E(G') \subseteq EG$.

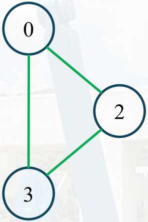
Induced Subgraph

A graph G' is an induced subgraph of G if G' is a subgraph of G and for any two vertices $u, v \in V(G')$, $(u, v) \in E(G)$ if and only if $(u, v) \in E(G')$.

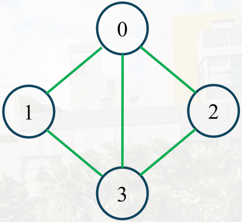
Examples



G



H_1



H_2

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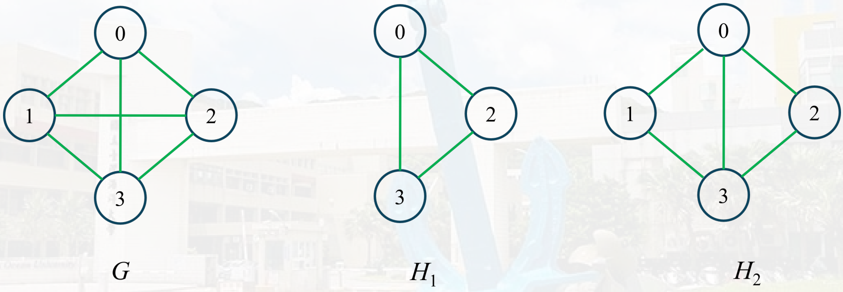

- H_1, H_2: subgraphs of G.

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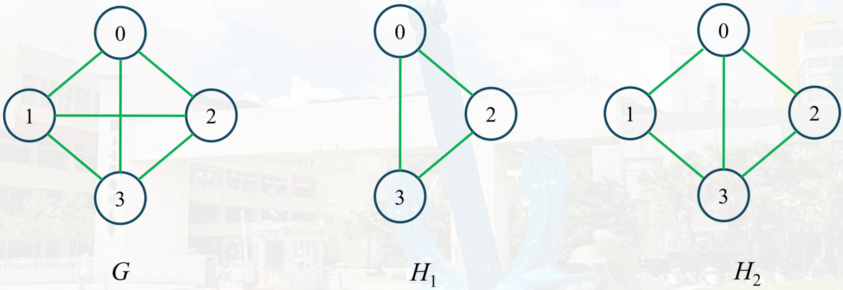
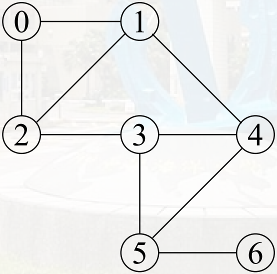


- H_1, H_2: subgraphs of G.
- H_1 is an induced subgraph of G, but H_2 is NOT.

Path (1/2)

Path

A (directed or undirected) path from vertex u to vertex v in graph G is a sequence of vertices $u, i_1, i_2, \ldots, i_k, v$, such that $(u, i_1), (i_1, i_2), \ldots, (i_k, v)$ are edges in $E(G)$.



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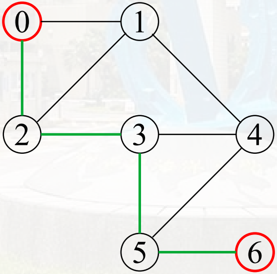

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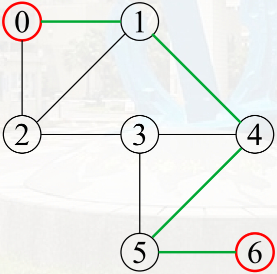

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- The length of a path is the number of edges on it.
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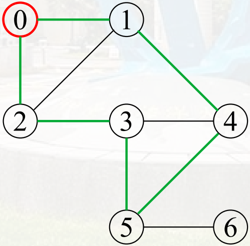
  - A simple path from v to v.



Connected and Connected Component

Connected

- In an undirected graph G, two vertices u and v are connected iff there is a path in G from u to v.
- An undirected graph is **connected** iff for every pair of district vertices u and v in $V(G)$ there is a path from u to v in G.

Connected Component

A **connected component** (or simply a component) H of an undirected graph is a maximal connected subgraph.

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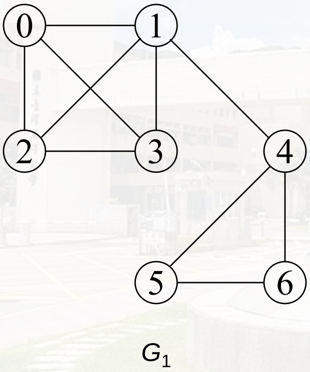
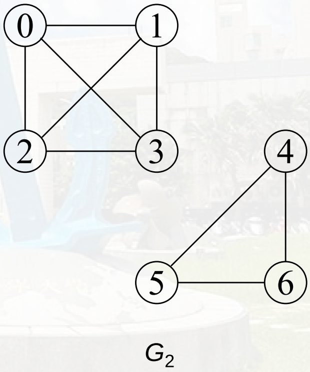
tree

A tree is a connected acyclic (i.e., has no cycles) graph.

Example of Connected Components





Strongly Connected Graph

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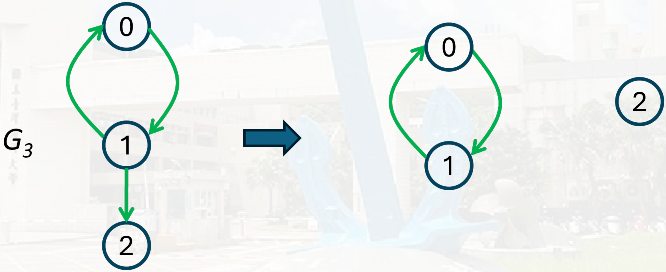
A strongly connected component is a maximal subgraph that is strongly connected.

Strongly Connected Components



Vertex Degree

- The degree of a vertex is the number of edges **incident to** that vertex.
- For a directed graph G,
  - The in-degree of a vertex is the number of edges for which vertex is head.
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Let d_i be the degree of vertex i in an n-vertex graph $G = (V, E)$, then

$$|E| = \frac{1}{2}\sum_{i=1}^{n} d_i.$$

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2 Graph Representations

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- Two most commonly used representation for a graph:
  - Adjacency Matrices
  - Adjacency Lists
- The choice of the representation:
  - the application
  - the functions one expects to perform on the graph
  - characteristics of the input graph

Adjacency Matrix

The adjacency matrix of an n-vertex graph G is a two-dimensional $n \times n$ array a, with the property that

- $a[i][j] = 1$ iff $(i, j) \in E(G)$;
- $a[i][j] = 0$ iff there is no such edge (i, j) in G.

Remark

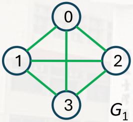
The adjacency matrix for an undirected graph is symmetric.

Adjacency Matrix (2/2)

- For an undirected graph the degree of any vertex i is its row sum.
- For an directed graph the row sum is its out-degree and the column sum is its in-degree.

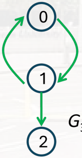
Adjacency Matrices (Examples)

The adjacency matrix of G_1



G_1

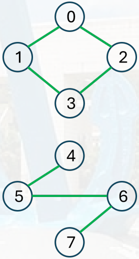
	0	1	2	3
0	0	1	1	1
1	1	0	1	1
2	1	1	0	1
3	1	1	1	0



G_3

	0	1	2
0	0	1	0
1	1	0	1
2	0	0	0

The adjacency matrix of G_3



G_4

	0	1	2	3	4	5	6	7
0	0	1	1	0	0	0	0	0
1	1	0	0	1	0	0	0	0
2	1	0	0	1	0	0	0	0
3	0	1	1	0	0	0	0	0
4	0	0	0	0	0	1	0	0
5	0	0	0	0	1	0	1	0
6	0	0	0	0	0	1	0	1
7	0	0	0	0	0	0	1	0

The adjacency matrix of G_4

Adjacency Lists

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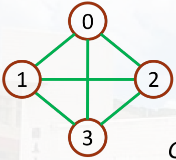
- The data field of a chain node stores the index of an adjacent vertex.

Adjacency Lists Examples



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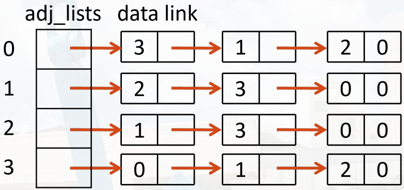
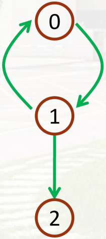
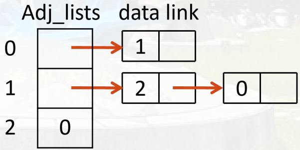


Adjacent list of G_1



G_3



Adjacent list of G_3

Remark: Weighted Edges

- In many applications, the edges of a graph have weights associated with them.

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- In many applications, the edges of a graph have weights associated with them.
    - importance, costs, distance, etc.
- The adjacency matrix entries `a[i][j]` would keep this information.
- When adjacency lists are used, we can introduce an additional field **weight** in the list nodes.

Discussions