Heaps

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Fall 2024

Outline

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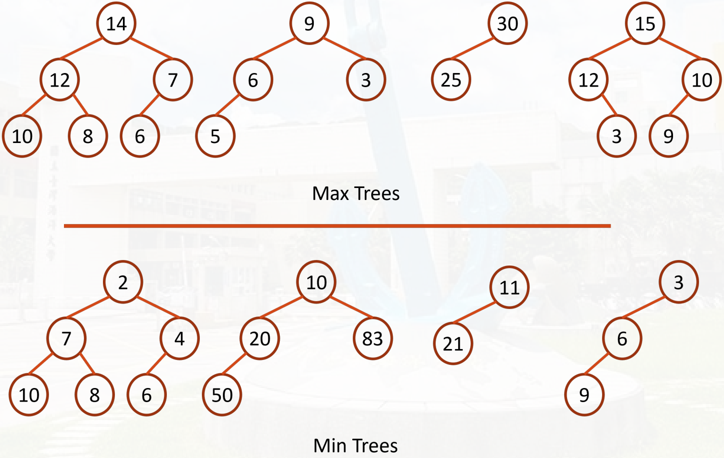
Max Heap

A complete binary tree that is also a max tree.

Min Heap

A complete binary tree that is also a min tree.

Examples: Max & Min Trees



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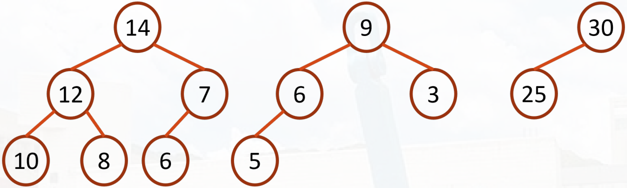
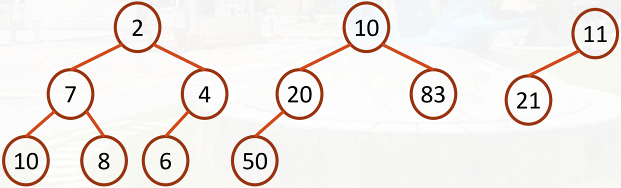


Max Heaps



Min Heaps

The Key Application: Priority Queues

- Heaps are frequently used to implement priority queues.
- In this kind of queue,
  - the element to be deleted is the one with highest (or lowest) priority.
  - at **any time**, an element with arbitrary priority can be **inserted** into the queue.

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- A heap is a complete binary tree.

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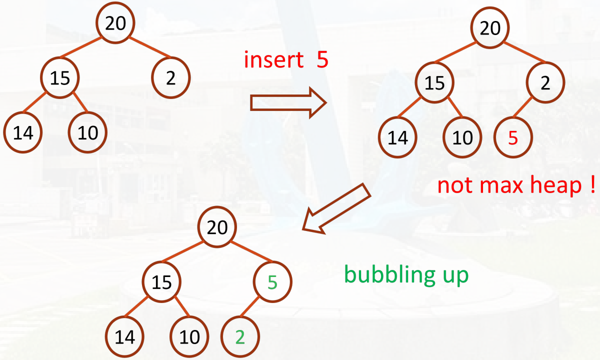

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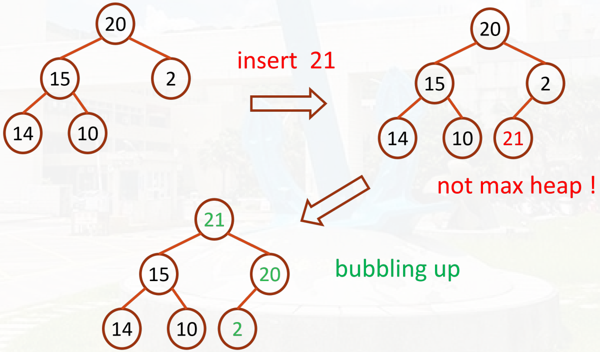

The Code for Insertion into a Max Heap

- Consider the following declarations:

```
#define MAX_ELEMENTS 200  /* maximum heap size+1 */
#define HEAP_FULL (n) (n == MAX_ELEMENTS -1)
#define HEAP_EMPTY (n)  (!n)
typedef struct {
    int key;
    /* other fields */
} element;
element heap[MAX_ELEMENTS];
int n = 0;
```

The Code for Insertion into a Max Heap

```
void push (element item, int *n) {
/* insert item into a max heap of current size *n */
    int i;
    if (HEAP_FULL(*n)) {
        printf("The heap is full.\n");
        exit(EXIT_FAILURE);
    } // O(1) time
    i = ++(*n);
    while ((i != 1) && (item.key > heap[i/2].key)) {
        heap[i] = heap[i/2];
        i /= 2;
    } //  O(lg n) time
    heap[i] = item; // O(1) time
}
```

- The time complexity of the insertion: $O(\lg n)$.

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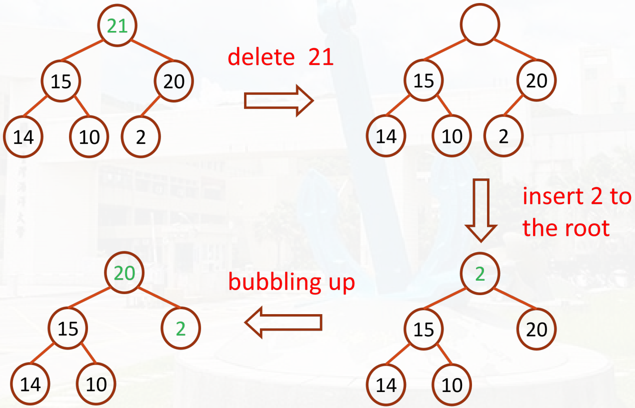
- When an element is to be deleted from a max heap, it is ALWAYS taken from the root of the heap.
- The steps of deletion from a Max heap:
  - delete the root node.
  - insert the last node into the root (say r).
  - use the bubbling up process to ensure that the resulting heap remains a max heap (a.k.a. heapify at r).

Illustration of Deletion from a Max Heap



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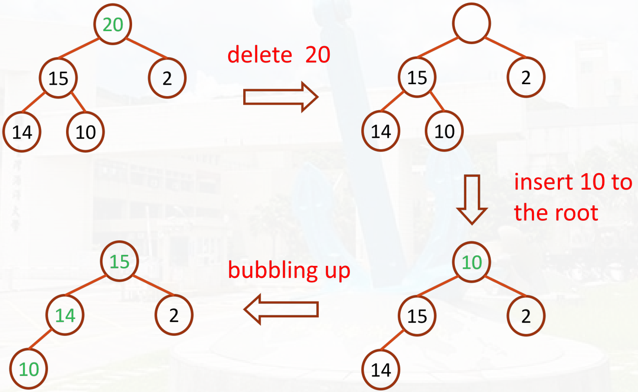

The Code for Deletion from a Max Heap

```
element pop(int *n) {
/* delete element with the highest key from the heap */
    int parent, child;
    element item, temp;
    if (HEAP_EMPTY(*n)) {
        fprintf(stderr, "The heap is empty\n");
        exit(EXIT_FAILURE);
    }
    /* save value of the element with the highest key */
    item = heap[1];
    /* use last element in heap to adjust heap */
    temp = heap[(*n)--];
    parent = 1;
    child = 2; // default: the left child
    while (child <= *n) { // O(lg n) time
    /* find the larger child of the current parent */
        if ((child < *n) && (heap[child].key < heap[child+1].key))
            child++; // okay, it's the right child!
        if (temp.key >= heap[child].key) break; // the new root is the maximum!
        /* if the max-child gets larger key, move to the next lower level */
        heap[parent] = heap[child];
        parent = child;
        child *= 2;
    }
    heap[parent] = temp;
    return item;
}
```

Time Complexity of the Deletion from a Max Heap

- Delete the root node: $O(1)$.
- Insert the last node to the root: $O(1)$.
- Since the height of the heap is $\lceil \lg(n+1)) \rceil$, the while loop is iterated for $O(\lg n)$ times.
- Thus, the overall time complexity: the time complexity of the deletion: $O(\log n)$.

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   - Building a heap

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An $O(n)$ time algorithm for building a (max) heap

Input: n numbers: $x_1, x_2, \ldots, x_n$.

Efficient Heap Construction

1. For each input number x_i, insert x_i into array A at $A[i-1]$ one by one.
2. For $i = \lfloor n/2 \rfloor - 1$ down to 0:
   - Run $\texttt{heapify}(A, i)$

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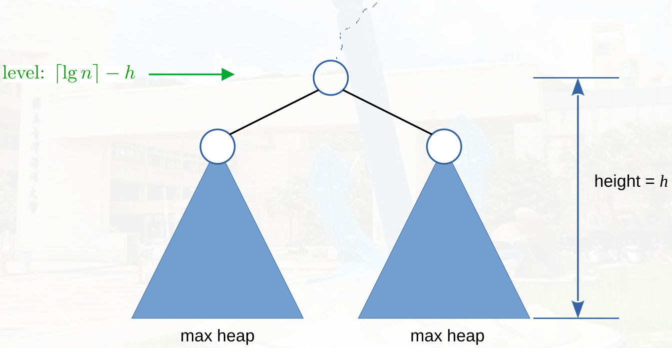
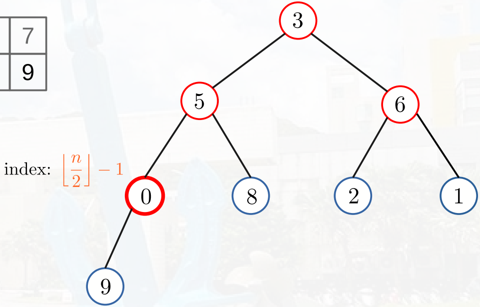
- That is, we build a heap in a bottom-up fashion!

Heap recursive view (bottom-up)



Nodes to be Heapified

index	0	1	2	3	4	5	6	7
	3	5	6	0	8	2	1	9



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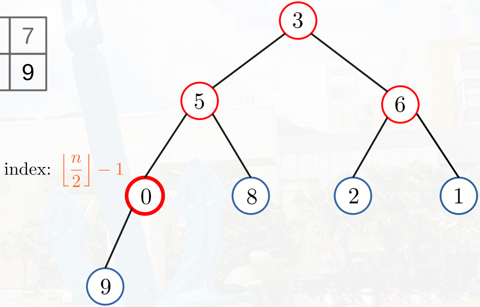

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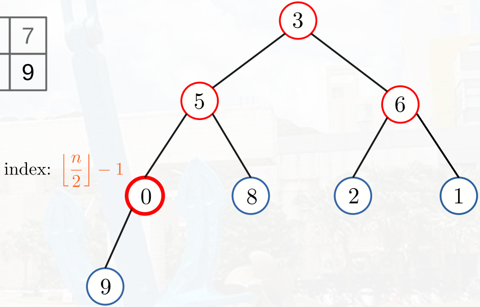

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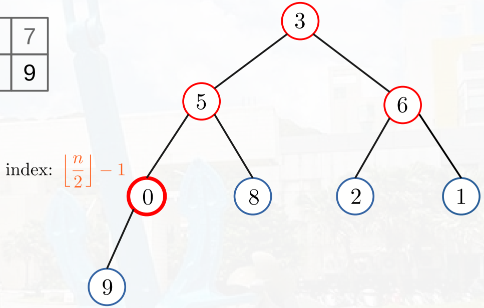

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  - n_h: the number of nodes at level h.

Discussions