Final Exam of Discrete Mathematics

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Note: Please list complete process of the calculation or the proof for each problem.

1. (a) For $A = \{1, 2, 3\}$, how many **antisymmetric** relations are there on A? (5%)
   (b) Let A be a set with $|A| = n$. How many **symmetric** relations are there on A? (5%)
2. For $n \in \mathbb{Z}^+$, $n \geq 2$, let $\phi(n)$ be the number of positive integers m where $1 \leq m < n$ and $\gcd(m, n) = 1$. This function is known as *Euler's phi function*.
   (a) Please calculate $\phi(100)$. (5%)
   (b) If $m, n \in \mathbb{Z}^+$, prove that $\phi(n^m) = n^{m-1}\phi(n)$. (10%).
3. (a) How many derangements are there for the integers $1, 2, 3, 4, 5$? (5%)
   (b) How many derangements are there for $1, 2, \ldots, n$? (Express your answer by a closed-form of '$\sum$') (5%)
   (c) Given $e^x = 1 + x + x^2/2! + x^3/3! + \ldots = \sum_{n=0}^{\infty} x^n/n!$. Please explain why the number of derangements for $1, 2, \ldots, 10000$ is approximately $10000! \cdot e^{-1}$. (5%)
4. (a) What is the generating function for the sequence $1, 1, 1, \ldots$? (5%)
   (b) What is the generating function for the sequence $0, 1, 2, 3, \ldots$? (5%)
   (c) What is the generating function for the sequence $0,\ 0,\ 1^2, 2^2, 3^2, 4^2, \ldots$? (10%)
5. (a) Please find out the coefficient of x^3 in $(1 - 2x)^{-7}$. (10%)
   (b) Use generating functions to determine how many three-element subsets of $S = \{1, 2, 3, \ldots, 10\}$ contain no consecutive integers. (10%)
6. Please solve the recurrence relation $a_n - 3a_{n-1} = 5 \cdot 3^n$, where $n \geq 1$ and $a_0 = 2$. (10%)
7. Please solve the recurrence relation $a_{n+2} - 4a_{n+1} + 4a_n = 2^n$, $n \geq 0$, $a_0 = 1$ and $a_1 = 2$. (10%)
8. (a) Find a recurrence relation for the number of binary strings of length n that do not contain consecutive 0's. (5%)
   (b) Solve the recurrence in (a). (10%)
9. A *Triple Tower of Hanoi* contains n disks (having different diameters) with holes in their centers. These disks can be stacked on any peg. The objective is to transfer the disks one at a time so that we end up with the original stack on peg 3. Each of pegs 1, 2, and 3 may be used as a temporary location for any disk(s), but at no time are we allowed to have a larger disk on top of a smaller one on any peg.
   (a) Let a_n be the minimum number of moves needed to do this for n disks, please give a recurrence relation for a_n. (5%)
   (b) Please solve the recurrence relation in (a). (10%)