Economics and Computation (2026 Fall)

Assignment 02

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Instructions. Show the essential reasoning and value calculations for every answer. When constructing an allocation, list every agent's bundle and verify the requested fairness property. Unless stated otherwise, all allocations are complete, and valuations for indivisible goods are nonnegative and additive. Give exact values whenever possible. The total score is 100 points.

Part I: Cake Cutting

Throughout Part I, the cake is $[0, 1]$, and valuations are nonnegative, additive, and divisible, with $V_i([0, 1]) = 1$. A density f_i represents $V_i(X) = \int_X f_i(x)\, dx$. Individual endpoints have value zero.

Problem 1. Dubins–Spanier on a small instance.

There are three agents. Their densities are constant on each of the following intervals:

	$[0, \frac{1}{3}]$	$[\frac{1}{3}, \frac{2}{3}]$	$[\frac{2}{3}, 1]$
f_1	2	1	0
f_2	0	2	1
f_3	1	0	2

Use the discrete Dubins–Spanier procedure from the lecture. At a stage with remaining cake $[x, 1]$, each active agent i marks the leftmost point y_i satisfying $V_i([x, y_i]) = \frac{1}{3}$. The agent with the leftmost mark receives that prefix and leaves. The final agent receives all remaining cake. Break ties between agents by smaller index.

(a) Calculate every active agent's mark at each stage. Give the three final pieces A_1, A_2, A_3.

(b) Compute the 3×3 table whose (i, j) entry is $V_i(A_j)$. Is the allocation proportional? Is it envy-free? Identify every ordered pair (i, j) for which agent i envies agent j.

Problem 2. Executing Selfridge–Conway.

There are three agents with the following piecewise constant densities:

	$[0, \frac{1}{3}]$	$[\frac{1}{3}, \frac{2}{3}]$	$[\frac{2}{3}, 1]$
f_1	1	1	1
f_2	$\frac{3}{2}$	1	$\frac{1}{2}$
f_3	$\frac{1}{2}$	1	$\frac{3}{2}$

Agent 1 begins by cutting

$$X_1 = [0, \tfrac{1}{3}], \qquad X_2 = [\tfrac{1}{3}, \tfrac{2}{3}], \qquad X_3 = [\tfrac{2}{3}, 1].$$

Agent 2 trims the *right end* of her favorite piece X_1. Write the trimming as $T = [t, \frac{1}{3}]$, so that $V_2(X_1 \setminus T) = V_2(X_2)$.

Follow the Selfridge–Conway rules: agent 3 chooses a main piece first. If agent 3 does not take the trimmed piece, agent 2 must take it; otherwise, agent 2 chooses from the two remaining main pieces. Agent 1 receives the last main piece.

Let $a \in \{2, 3\}$ be the recipient of the trimmed main piece, and let a' be the other agent in $\{2, 3\}$. Agent a' cuts T into three pieces of equal value to herself. These pieces are chosen in the order a, then agent 1, then a'. Whenever an agent is indifferent between available intervals, she takes the leftmost one.

(a) Find t and determine who receives each main piece. Identify a and a'.

(b) Find the two cut points within T and determine who receives each trimming piece. Express every final piece A_i as an interval or a union of intervals.

(c) Compute the full table of values $V_i(A_j)$ and verify that the final allocation is envy-free.

Problem 3. Query complexity and the Even–Paz guarantee.

Let $n = 2^h$, where $h \geq 1$. In Even–Paz, a group S with current interval $[y, z]$ asks each member i to mark a point x_i satisfying $V_i([y, x_i]) = \frac{1}{2}V_i([y, z])$. Sort the marks and split the interval at a point between the two middle marks. Recurse with the left half of the agents on the left subcake, and with the right half on the right subcake. A singleton receives its whole subcake. Break mark ties by agent index.

(a) In discrete Dubins–Spanier, every active agent makes one cut query at each stage with $k = n, n-1, \ldots, 2$ agents. Find the exact total number of cut queries and its asymptotic order.

(b) Let $T(n)$ count the cut queries used by Even–Paz, with $T(1) = 0$. Write and solve its recurrence. Explain how evaluation queries can supply the values of current subcakes, and give the asymptotic total number of Robertson–Webb queries.

(c) Briefly justify that, at depth d, every active agent values her current subcake at least 2^{-d}. Deduce proportionality.

Problem 4. Optimal proportional cake cutting.

There are two agents. Their densities on $L = [0, \frac{1}{2}]$ and $R = [\frac{1}{2}, 1]$ are

	L	R
f_1	2	0
f_2	$\frac{3}{2}$	$\frac{1}{2}$

Let x and y be the fractions of the *lengths* of L and R, respectively, allocated to agent 1. Agent 2 receives the remaining fractions. Write $\mathrm{SW}(A) = V_1(A_1) + V_2(A_2)$.

(a) Express both agents' utilities in terms of x, y. Formulate a linear program maximizing social welfare subject to proportionality and $0 \leq x, y \leq 1$.

(b) Solve the program without using an LP solver. Give an optimal allocation using explicit intervals, and verify that it is also envy-free.

(c) Find the unconstrained welfare optimum and compute

$$\rho_{\mathrm{PROP}}(I) = \frac{\max_A \mathrm{SW}(A)}{\max_{A \text{ proportional}} \mathrm{SW}(A)}.$$

What lower bound on the global price of proportionality follows from this instance?

Part II: Fair Allocation of Indivisible Goods

Write $v_i(S) = \sum_{g \in S} v_i(g)$. Use the lecture's EFX convention: the required inequality must hold after removing *every* good from the other agent's bundle.

Problem 5. Checking EF, EF1, and EFX.

Two agents have the following values:

	g_1	g_2	g_3	g_4
v_1	4	5	2	1
v_2	1	4	3	2

Consider these allocations; the first bundle in each pair belongs to agent 1:

$$A = (\{g_1\}, \{g_2, g_3, g_4\}),$$
$$B = (\{g_1, g_4\}, \{g_2, g_3\}),$$
$$C = (\{g_1, g_3\}, \{g_2, g_4\}).$$

(a) For each allocation, determine whether it is EF, EF1, and EFX. Present your nine answers in a table. Support each answer with the relevant comparisons; for every failed property, give a specific violating pair and, when relevant, a removed good.

(b) Find the largest $\alpha \in [0, 1]$ for which A is α-EFX, meaning $v_i(A_i) \geq \alpha\, v_i(A_j \setminus \{g\})$ for every i, j and every $g \in A_j$.

Problem 6. Constructing EF1 and EFX allocations.

There are three agents and six goods:

	g_1	g_2	g_3	g_4	g_5	g_6
v_1	12	8	5	4	2	1
v_2	12	8	6	3	2	1
v_3	10	3	9	6	3	1

(a) Find a complete allocation that is EF1 but not EFX. Verify EF1 and identify a pair (i, j) and good $g \in A_j$ witnessing the failure of EFX. You may use round-robin; if so, report the picking order.

(b) Find a complete EFX allocation for the same instance. Verify EFX for every ordered pair of distinct agents.

Problem 7. An envy-cycle elimination step.

Three agents have values

	a	b	c	d
v_1	1	4	2	3
v_2	2	1	4	6
v_3	4	2	1	6

Start from the partial allocation $A_1 = \{a\}$, $A_2 = \{b\}$, $A_3 = \{c\}$; good d is unallocated. The envy graph has an edge $i \to j$ exactly when $v_i(A_i) < v_i(A_j)$. A source has indegree zero.

(a) Draw the initial envy graph. Eliminate the cycle $1 \to 2 \to 3 \to 1$ by simultaneously replacing (A_1, A_2, A_3) with (A_2, A_3, A_1). Give the new bundles.

(b) Recompute the envy graph and identify all sources. Give d to the source with the smallest index. Draw the resulting graph.

(c) Verify that the resulting complete allocation is EF1. More generally, explain why giving one unallocated good to a source preserves EF1 whenever the current partial allocation is EF1.

Problem 8. Computing and attaining maximin shares.

Three agents value five goods as follows:

	g_1	g_2	g_3	g_4	g_5
v_1	8	7	6	5	4
v_2	7	8	5	6	4
v_3	8	6	7	4	5

Recall

$$\mu_i^3(M) = \max_{(P_1,P_2,P_3) \text{ partition of } M} \min_{k \in \{1,2,3\}} v_i(P_k).$$

An MMS allocation gives every agent i value at least $\mu_i^3(M)$.

(a) Compute $\mu_i^3(M)$ for each agent. For each value, give a partition attaining it and prove a matching upper bound.

(b) Find a complete MMS allocation. List all bundles and verify the three MMS guarantees.

(c) Does a proportional allocation exist for this instance? Prove your answer.

Problem 9. EF1 and proportionality up to one good.

An allocation is PROP1 if, for every agent i, either $v_i(A_i) \geq v_i(M)/n$, or there exists a good $g \in M \setminus A_i$ such that

$$v_i(A_i) + v_i(g) \geq \frac{v_i(M)}{n}.$$

(a) Prove that every EF1 allocation is PROP1 for nonnegative additive valuations. *Hint.* For a fixed agent i, use EF1 to bound her value for each other bundle, and add the inequalities. Consider a most valuable good outside A_i.

(b) Construct an instance with two agents and three goods, with strictly positive additive values, and a complete allocation that is PROP1 but not EF1. Give the valuation table, the allocation, and the comparisons proving both claims.

Problem 10. Nash social welfare and Pareto optimality.

Two agents have values

	a	b	c
v_1	8	7	1
v_2	6	3	2

Maximizing Nash social welfare is equivalent to maximizing the Nash product

$$v_1(A_1)v_2(A_2).$$

(a) Find all allocations that maximize this product. Show your calculations over all possible bundles $A_1 \subseteq \{a, b, c\}$; agent 2 receives the complement.

(b) Determine whether the maximizing allocation is EF1 and whether it is EFX. Verify your answers.

(c) Prove that, in any finite indivisible-goods instance, an allocation that maximizes the Nash product and has strictly positive product is Pareto optimal. Explain why positivity matters to this product-comparison argument, and apply the result to part (a).

Problem 11. The p-mean and Nash social welfare.

Let $u_i > 0$ be agent i's utility, for $i = 1, 2, \ldots, n$; for example, $u_i = v_i(A_i)$ for an allocation $A = (A_1, A_2, \ldots, A_n)$. For every $p \in \mathbb{R} \setminus \{0\}$, define the p-mean welfare by

$$W_p = \left(\frac{1}{n} \sum_{i=1}^{n} u_i^p \right)^{1/p}.$$

Prove that W_p converges to the normalized Nash social welfare as $p \to 0$; that is, show that

$$\lim_{p \to 0} W_p = \left(\prod_{i=1}^{n} u_i \right)^{1/n}.$$

Hint. Take logarithms and apply L'Hôpital's rule.