Equilibrium Concepts

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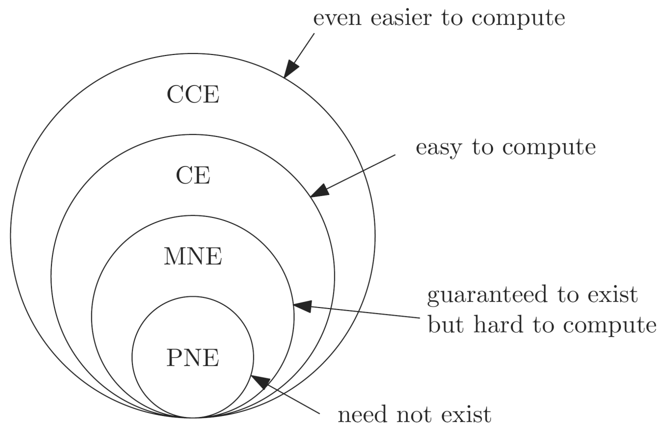
Outline

A hierarchy of equilibrium concepts



Outline

Cost-Minimization Games

A cost-minimization game has the following ingredients:

- a finite number of k agents;
- a finite set S_i of pure strategies for each agent i;
- a nonnegative cost function $C_i(\mathbf{s})$ for each agent i.
  - $\mathbf{s} \in S_1 \times S_2 \times \cdots \times S_k$: a strategy profile or outcome.

For example, the network creation game.

Payoff-Maximization Games

A **payoff-maximization** game has the following ingredients:

- a finite number of k agents;
- a finite set S_i of pure strategies for each agent i;
- a nonnegative payoff function $\pi_i(\mathbf{s})$ for each agent i.
  - $\mathbf{s} \in S_1 \times S_2 \times \cdots \times S_k$: a strategy profile or outcome.

For example, the Rock-Paper-Scissors game, two-party election game, etc.

Outline

1. Cost Minimization and Payoff Maximization
2. Pure Nash Equilibria (PNE)
3. Mixed Nash Equilibria (MNE)
4. Correlated Equilibria (CE)
5. Coarse Correlated Equilibria (CCE)
6. Appendix: Network Creation Games

Pure Nash Equilibrium (PNE)

Pure Nash Equilibrium (PNE)

A strategy profile **s** of a cost-minimization game is a pure Nash equilibrium (PNE) if for every agent $i \in \{1, 2, \ldots, k\}$ and every unilateral deviation $s_i' \in S_i$,

$$C_i(\mathbf{s}) \leq C_i(s_i', \mathbf{s}_{-i}).$$

- $\mathbf{s}_{-i}$: the vector **s** with the ith component removed.

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Mixed Nash Equilibrium (MNE)

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Distributions $\sigma_1, \ldots, \sigma_k$, over strategy sets $S_1, \ldots, S_k$ respectively, of a cost-minimization game constitute a mixed Nash equilibrium (MNE) if for every agent $i \in \{1, 2, \ldots, k\}$ and every unilateral deviation $s_i' \in S_i$,

$$\mathbf{E}_{\mathbf{s} \sim \sigma}[C_i(\mathbf{s})] \leq \mathbf{E}_{\mathbf{s} \sim \sigma}[C_i(s_i', \mathbf{s}_{-i})].$$

- σ: the product distribution $\sigma_1 \times \cdots \times \sigma_k$.

Product of Mixed Strategies

Player 2

			q_1 rock	q_2 scissors	q_3 paper
Player 1	p_1	rock	$p_1 q_1$ 0, 0	$p_1 q_2$ 1, −1	$p_1 q_3$ −1, 1
	p_2	scissors	$p_2 q_1$ −1, 1	$p_2 q_2$ 0, 0	$p_2 q_3$ 1, −1
	p_3	paper	$p_3 q_1$ 1, −1	$p_3 q_2$ −1, 1	$p_3 q_3$ 0, 0

$q_1, q_2, q_3 \longrightarrow$ probabilities

$p_1, p_2, p_3 \downarrow$ probabilities

$$\begin{cases} p_1 + p_2 + p_3 = 1. \\ q_1 + q_2 + q_3 = 1. \end{cases}$$

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Correlated Equilibrium (CE)

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A distribution σ on the set $S_1 \times \cdots \times S_k$ of outcomes of a cost-minimization game is a correlated equilibrium (CE) if for every agent $i \in \{1, 2, \ldots, k\}$ and every unilateral deviation $s_i' \in S_i$,

$$\mathbf{E}_{\mathbf{s} \sim \sigma}[C_i(\mathbf{s}) \mid s_i] \leq \mathbf{E}_{\mathbf{s} \sim \sigma}[C_i(s_i', \mathbf{s}_{-i}) \mid s_i].$$

Stop or Go?

Matrix of costs

	Stop	Go
Stop	1, 1	1, 0
Go	0, 1	5, 5

- Two PNEs.

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Matrix of costs

	Stop	Go
Stop	prob. $= 0$ 1, 1	prob. $= 1/2$ 1, 0
Go	prob. $= 1/2$ 0, 1	prob. $= 0$ 5, 5

- A CE for example.
- Cannot correspond to a MNE.

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Game of Chicken

- A.k.a. Hawk-Dove Game.
  - A model of conflict for two players.

	Dare	Chicken
Dare	0, 0	7, 2
Chicken	2, 7	6, 6

- Two PNE & One MNE.
- The expected utility of each player in the MNE: $\frac{1}{3} \cdot \frac{2}{3} \cdot 7 + \frac{2}{3} \cdot \frac{1}{3} \cdot 2 + \frac{2}{3} \cdot \frac{2}{3} \cdot 6 = \frac{14}{3}$.

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Game of Chicken

- A correlated equilibrium.
  - Check that it is an equilibrium if a player is assigned “Dare”.
  - Check that it is an equilibrium if a player is assigned “Chicken Out”.

	Dare	Chicken
Dare	$\text{prob.} = 0$ 0, 0	$\text{prob.} = 1/3$ 7, 2
Chicken	$\text{prob.} = 1/3$ 2, 7	$\text{prob.} = 1/3$ 6, 6

- The expected utility for each player: $7 \cdot (1/3) + 2 \cdot (1/3) + 6 \cdot (1/3) = 5$.

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$$\mathbf{E}_{\mathbf{s}\sim\sigma}[C_i(\mathbf{s})] \leq \mathbf{E}_{\mathbf{s}\sim\sigma}[C_i(s_i', \mathbf{s}_{-i})].$$

CE $\subseteq$ CCE?

$$\begin{aligned}
\mathbf{E}_{\mathbf{s}\sim\sigma}[C_i(\mathbf{s})] &= \sum_{a\in S_i} \mathbf{E}_{\mathbf{s}\sim\sigma}[C_i(\mathbf{s}) \mid s_i = a] \Pr[s_i = a] \\
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\end{aligned}$$

CCE Example

	A	B	C
A	prob. $= 1/3$ $1, 1$	$-1, -1$	$0, 0$
B	$-1, -1$	prob. $= 1/3$ $1, 1$	$0, 0$
C	$0, 0$	$0, 0$	prob. $= 1/3$ $-1.1, -1.1$

- The payoff for each player (playing according to this distribution): $\frac{1}{3} \cdot 1 + \frac{1}{3} \cdot 1 - \frac{1}{3} \cdot 1.1 = 0.3$.
- A player playing fixed A or B while the opponent randomized according to this distribution: $\frac{1}{3} \cdot 1 - \frac{1}{3} \cdot 1 + \frac{1}{3} \cdot 0 = 0$.
- A player playing fixed C while the opponent randomized according to this distribution: $\frac{1}{3} \cdot 0 + \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot (-1.1) < 0$.

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- A player playing fixed C and the strategy profile follows this distribution:
  $\Rightarrow$ deviation is possible.
  - Not a CE.

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C	0, 0	0, 0	prob. $= 1/3$ −1.1, −1.1

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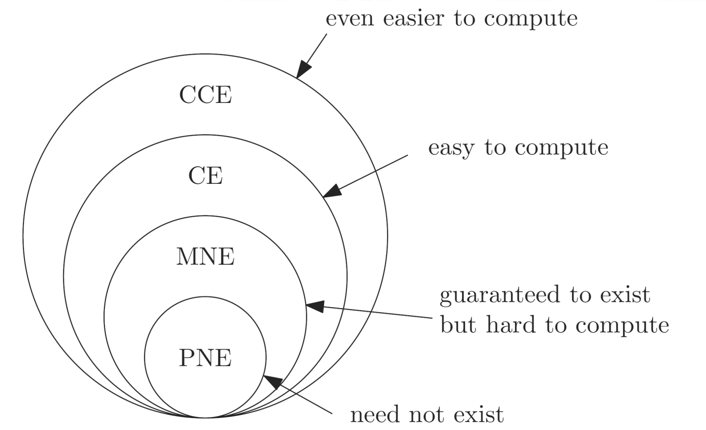
A hierarchy of equilibrium concepts



Outline

Network creation games

- First introduced in PODC 2003.


Alex Fabrikant


Ankur Luthra


Elitza Maneva


Christos H. Papadimitriou


Scott Shenker

Network creation games [Fabrikant et al. @PODC 2003]

- n players: 1, 2, ..., n.
- s_i: specified by a subset of $\{1, 2, \ldots, n\} \setminus \{i\} = [n] \setminus \{i\}$ as the strategy of player i.
  - The set of neighbors where player i forms a link (edge).
- G_s: the undirected graph with vertex set $[n]$ and edges corresponding to $s = \langle s_1, s_2, \ldots, s_n \rangle$.
- G_s has an edge $\{i, j\}$ if either $i \in s_j$ or $j \in s_i$.
- $d_s(i, j)$: the distance between i and j in G_s.
- G_s: an equilibrium graph (when the context is clear).

Network creation games (Two models)

The sum model

$$c_i(s) = \alpha|s_i| + \sum_{j=1}^{n} d_s(i,j).$$

The max model

$$c_i(s) = \alpha|s_i| + \max_{j=1}^{n} d_s(i,j).$$

- The total cost is $c(s) = \sum_{i=1}^{n} c_i(s)$.

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Network creation games (contd.)

Theorem [Fabrikant et al.@PODC 2003]

The PoA for the sum network creation game is $O(\sqrt{\alpha})$ for all α.

Preliminaries

Let's have a look at Fabrikant's results for $\alpha < 2$.

- $\alpha < 1$:
  - the social optimum: the complete graph.
  - ⋆ It's also a NE ($\therefore$ PoA $= 1$).

Preliminaries (contd.)

- $1 \le \alpha < 2$:
  - The social optimum: still the complete graph (i.e., K_n).
  - Any NE must be connected and has diameter ≤ 2.
  - ⋆ K_n is NOT a NE.
  - ⋆ The worst NE: a star.
    - $\alpha \cdot |E| + |E| \cdot 2 \cdot 1 + (\binom{n}{2} - |E|) \cdot 2 \cdot 2 = (\alpha - 2) \cdot |E| + 2$ [illegible]

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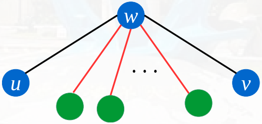

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    - $\alpha \cdot |E| + |E| \cdot 2 \cdot 1 + \left(\binom{n}{2} - |E|\right) \cdot 2 \cdot 2 = (\alpha - 2) \cdot |E| + 2n(n-1)$.
  - $$\begin{aligned} \text{PoA} &= \frac{C(\text{star})}{C(K_n)} = \frac{(\alpha-2)\cdot(n-1) + 2n(n-1)}{\alpha\binom{n}{2} + 2\cdot\binom{n}{2}\cdot 1} \\ &= \frac{4}{2+\alpha} - \frac{4-2\alpha}{n(2+\alpha)} \\ &< \frac{4}{3}. \end{aligned}$$

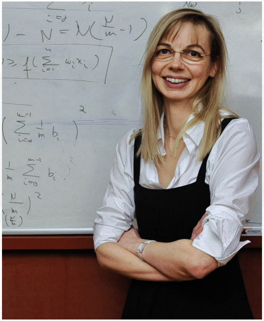
Preliminaries (contd.)

Lemma 1 [Albers et al. @SODA 2006]

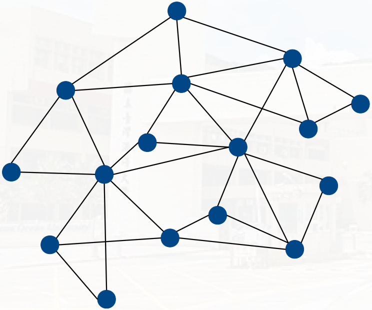
For any Nash equilibrium s and any vertex v_0 in G_s,

$$c(s) \leq 2\alpha(n-1) + n \cdot \mathrm{Dist}(v_0) + (n-1)^2.$$

- $\mathrm{Dist}(v_0) = \sum_{v \in V(G_s)} d_s(v_0, v)$.

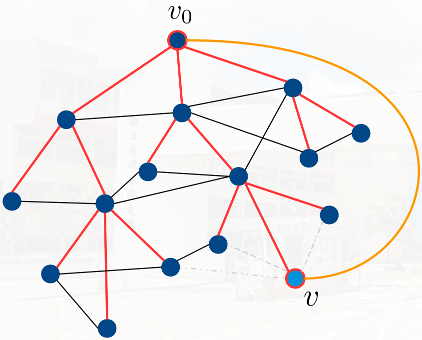


Sketch of proving Lemma 1



- A graph G_s corresponding to a NE s.

Sketch of proving Lemma 1



- $T(v_0)$: the shortest-path tree rooted at v_0.
- η_v: the number of tree edges built by v in $T(v_0)$.
  - ⋆ $c_v(s) \leq \alpha(\eta_v + 1) + \mathrm{Dist}(v_0) + n - 1$.

    $c_{v_0}(s) = \alpha \cdot \eta_{v_0} + \mathrm{Dist}(v_0)$.
- $c(s) = \sum_{v \in V(G_s) \setminus \{v_0\}} c_v(s) + c_{v_0}(s)$
  $\leq 2\alpha(n-1) + n \cdot \mathrm{Dist}(v_0) + (n-1)^2$.

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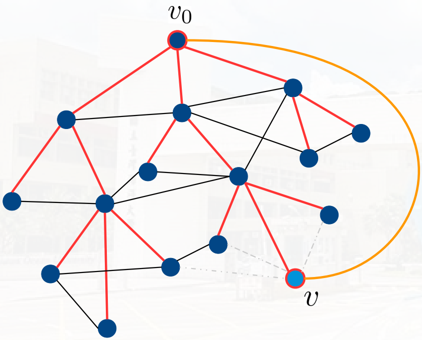

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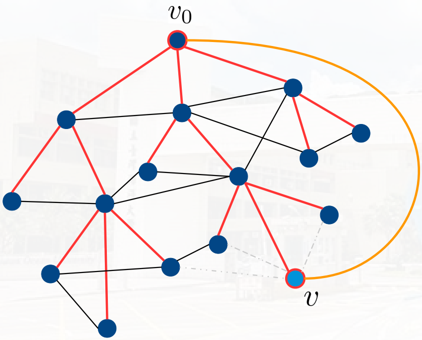

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$\leq 2\alpha(n-1) + n \cdot \mathrm{Dist}(v_0) + (n-1)^2$.

Preliminaries (contd.)

Lemma 2

If the shortest-path tree in an equilibrium graph G_s rooted at u has depth d, then $\text{PoA} \leq d + 1$.

- For some $u \in V$,

$$\begin{aligned}
\text{PoA} &\leq \frac{2\alpha(n-1) + n \cdot \text{Dist}(u) + (n-1)^2}{\alpha(n-1) + n(n-1)} \\
&\leq \frac{2\alpha(n-1) + n \cdot (n-1)d + (n-1)^2}{\alpha(n-1) + n(n-1)} \\
&< \frac{2\alpha(n-1) + n(n-1)(d+1)}{\alpha(n-1) + n(n-1)} \\
&\leq \max\left\{\frac{2\alpha(n-1)}{\alpha(n-1)}, \frac{n(n-1)(d+1))}{n(n-1)}\right\} \\
&= \max\{2, d+1\}.
\end{aligned}$$

Preliminaries (contd.)

Lemma 2

If the shortest-path tree in an equilibrium graph G_s rooted at u has depth d, then PoA $\leq d + 1$.

- For some $u \in V$,

$$\begin{aligned} \text{PoA} &\leq \frac{2\alpha(n-1) + n \cdot \text{Dist}(u) + (n-1)^2}{\alpha(n-1) + n(n-1)} \\ &\leq \frac{2\alpha(n-1) + n \cdot (n-1)d + (n-1)^2}{\alpha(n-1) + n(n-1)} \\ &< \frac{2\alpha(n-1) + n(n-1)(d+1)}{\alpha(n-1) + n(n-1)} \\ &\leq \max\left\{\frac{2\alpha(n-1)}{\alpha(n-1)}, \frac{n(n-1)(d+1))}{n(n-1)}\right\} \\ &= \max\{2, d+1\}. \end{aligned}$$

Preliminaries (contd.)

Lemma 2

If the shortest-path tree in an equilibrium graph G_s rooted at u has depth d, then PoA $\leq d + 1$.

- For some $u \in V$,

$$\begin{aligned}
\text{PoA} &\leq \frac{2\alpha(n-1) + n \cdot \text{Dist}(u) + (n-1)^2}{\alpha(n-1) + n(n-1)} \\
&\leq \frac{2\alpha(n-1) + n \cdot {\color{red}(n-1)d} + (n-1)^2}{\alpha(n-1) + n(n-1)} \\
&< \frac{2\alpha(n-1) + n(n-1)(d+1)}{\alpha(n-1) + n(n-1)} \\
&\leq \max\left\{\frac{2\alpha(n-1)}{\alpha(n-1)}, \frac{n(n-1)(d+1))}{n(n-1)}\right\} \\
&= \max\{2, d+1\}.
\end{aligned}$$