Preliminaries in Game Theory

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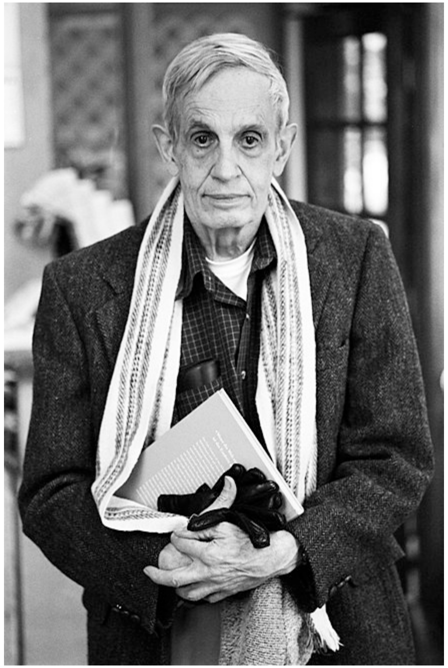
John Forbes Nash Jr. (1928–2015)

- American mathematician.
- Fundamental contributions to game theory.
- **Nobel Memorial Prize** in Economic Sciences with game theorists Reinhard Selten and John Harsanyi.
- **Abel Prize** with Louis Nirenberg for his work on nonlinear partial differential equations.

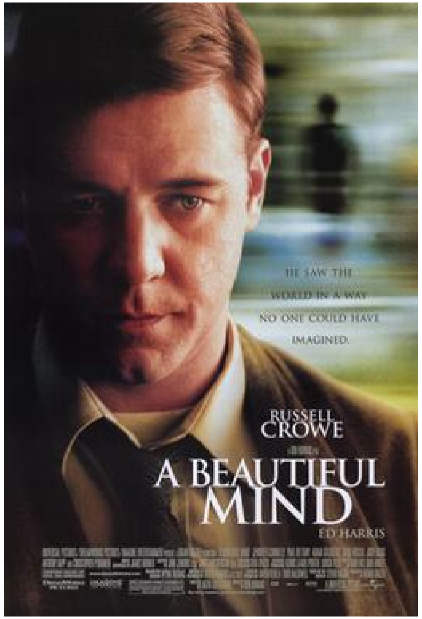


Nash in 2006.

Reference: https://en.wikipedia.org/wiki/John_Forbes_Nash_Jr.

A classic scene of “A Beautiful Mind”

- https://www.youtube.com/watch?v=2d_dtTZQyUM



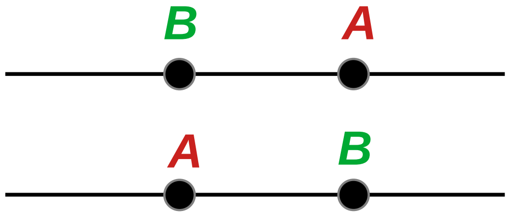
Starring: Russel Crowe

Before introducing **Nash Equilibria**...

- Let’s play around several “games” first.

Number Guessing

- Let’s say I have chosen a secret number ***A*** in my mind, which is among 1 and 100.
- Please guess it by a number ***B***.
- If $B < A$, I will tell you “Larger, please”.
- If $B > A$, I will tell you “Smaller, please”.



- How many times do you think you can find out this secret number?

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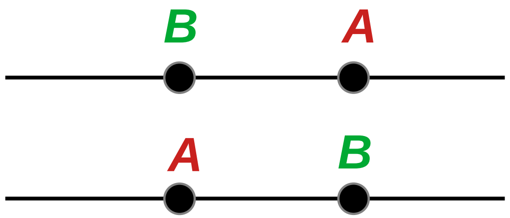
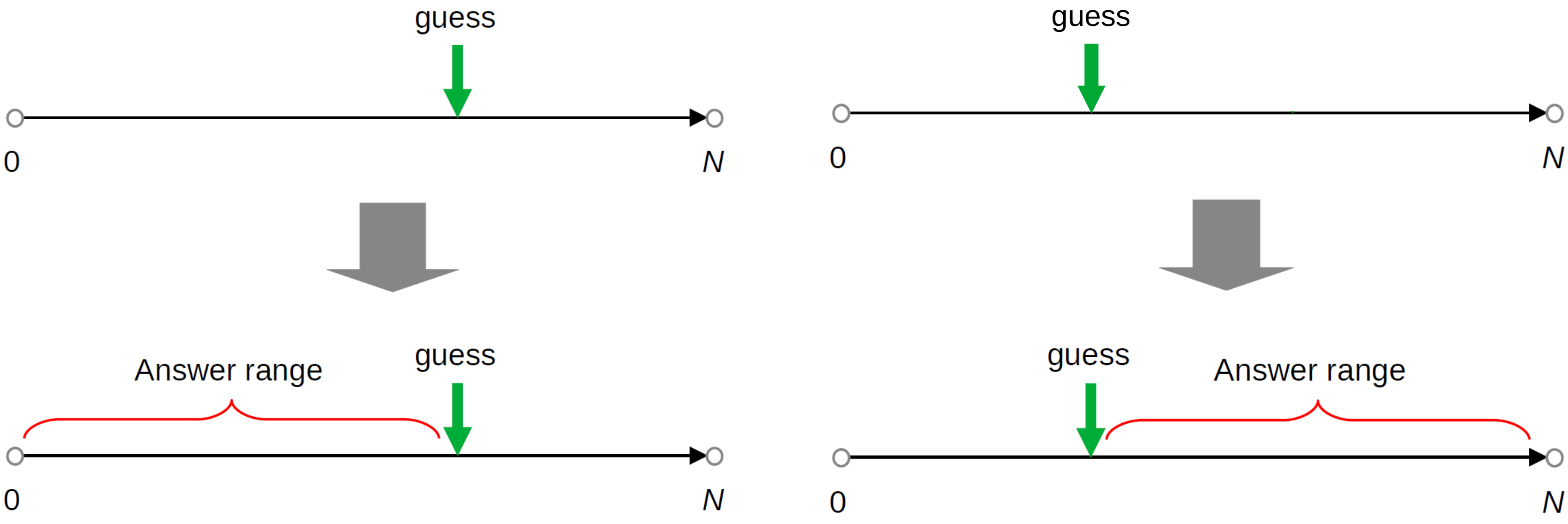


- How many times do you think you can find out this secret number? **Let's play to feel the strategic behaviors.**

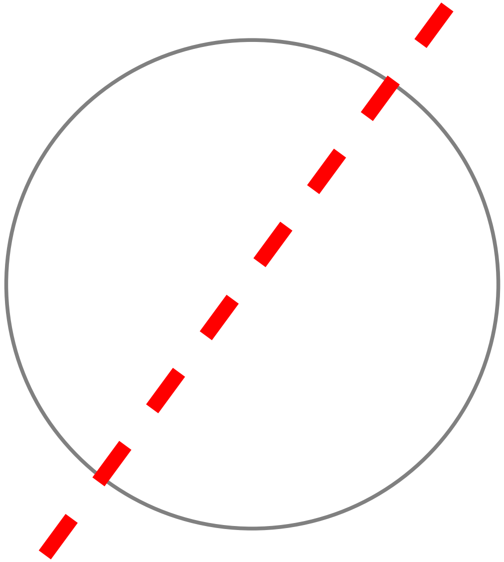
Adversarial Number Guessing

- The **demo** code.



Envy-Free Cake-Cutting

- Children wants everything to be FAIR.
- Actually, in their world, nothing is FAIR…… lol



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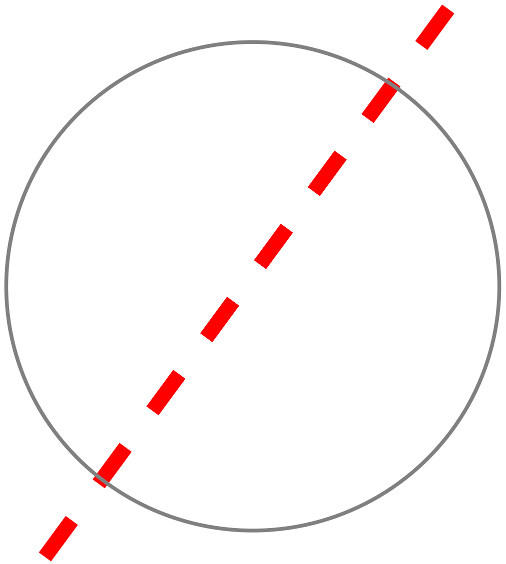
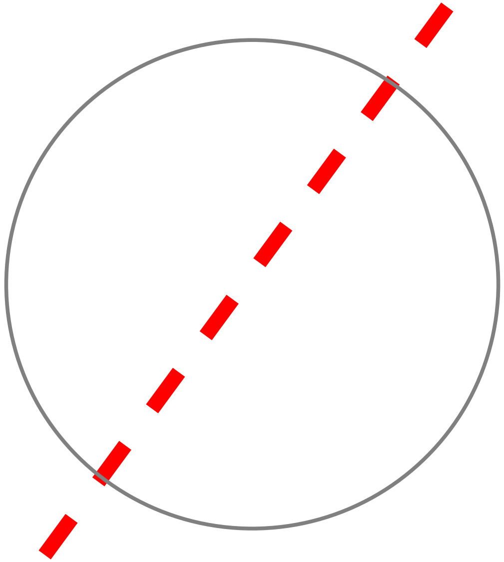
- Let’s say we want two kids to share a cake.



Envy-Free Cake-Cutting

- Children wants everything to be FAIR.
- Actually, in their world, nothing is FAIR…… lol
- Let’s say we want two kids to share a cake.
- Can you propose a way of cutting a cake so that two kids share a cake so that **no one envies the other?**



Prisoners’ Dilemma

- It was originally framed by Merrill Flood and Melvin Dresher in 1950.
- Let’s say there are two guys, ***A*** and ***B***, who broke into a luxury store and stole a treasure.

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- They had hided the treasure before the police caught them.

source

Prisoners' Dilemma

- It was originally framed by Merrill Flood and Melvin Dresher in 1950.
- Let's say there are two guys, ***A*** and ***B***, who broke into a luxury store and stole a treasure.
- They had hided the treasure before the police caught them.
- They were kept in two separated rooms.
  - That means, they cannot communicate with each other…
- Each of them was offered two choices: **Denial** or **confession**.

Prisoners’ Dilemma

- They were told that:

Prisoners’ Dilemma

- They were told that:
  - If both of them deny the fact of stealing the treasure, they will **BOTH** be sentenced in prison for **one** month.
  - If one of them confesses while the other one denies, the former will be set **FREE** while the latter will be sentenced in prison for **9** months.
  - If both confess, then they will both get **6** months in prison.
    - Because the police officers have got their images from the monitor...

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    - Because the police officers have got their images from the monitor…

- In your opinion, what should they do?
  - They cannot communicate, and they must make their decisions simultaneously.

Prisoners’ Dilemma

- We can use a “matrix” to formulate this game.
- Two **players**, two **actions** for each.
- If you are criminal *A*, what will you do?
- What’s the **solution** (outcome)?

Criminal *B*

Criminal *A*

	Denial	Confess
Denial	−1, −1	−9, 0
Confess	0, −9	−6, −6

Prisoners’ Dilemma

- **Dominant strategy?**

- **Socially inefficient.**
  - **Why is it inefficient?**

- **Price of Anarchy (PoA).**

Criminal *B*

Criminal *A*	Denial	Confess
Denial	−1, −1	−9, 0
Confess	0, −9	−6, −6

Bach or Stravinsky (BoS)

- A historical two-player game.
  - The battle of sexes (in *Games and Decisions* by Luce and Raiffa, 1957).
  - Say Amy and Bob want to pick a concert to go to.

Bach or Stravinsky (BoS)

- A historical two-player game.
  - The battle of sexes (in *Games and Decisions* by Luce and Raiffa, 1957).
  - Say Amy and Bob want to pick a concert to go to.
  - Both prefer to go together than to go home.
  - However, Amy prefers Bach while Bob prefers Stravinsky.

Bach or Stravinsky (BoS)

- What are the SOLUTIONS of the game?
- Is there any **dominant** strategy for either Amy or Bob?

		Bob	
		Bach	Stravinsky
Amy	Bach	2, 1	0, 0
	Stravinsky	0, 0	1, 2

The matrix form

Battle of Sexes (BoS)

- What are the SOLUTIONS of the game?
- Is there any **dominant** strategy for either Amy or Bob?

		Bob	
		Movie	Baseball Game
Amy	Movie	2, 1	0, 0
	Baseball Game	0, 0	1, 2

↑

The matrix form

Matching Pennies

- Two players, playing a game by throwing a penny.
- Both ‘heads’ or both ‘tails’: player 1 keeps both pennies.
- Otherwise, player 2 keeps both pennies.

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The matrix form →

		Player 2	
		head	tail
Player 1	head	1, −1	−1, 1
	tail	−1, 1	1, −1

Matching Pennies

- **Zero-sum**?
- Do dominant strategies exist?
- What are the solutions?

		Player 2	
		head	tail
Player 1	head	1, −1	−1, 1
	tail	−1, 1	1, −1

Rock-Scissors-Paper Game

rock

scissors

paper

Player 2

Player 1

	rock	scissors	paper
rock	0, 0	1, −1	−1, 1
scissors	−1, 1	0, 0	1, −1
paper	1, −1	−1, 1	0, 0

Rock-Scissors-Paper Game

- **Zero-sum**?
- **Dominant strategies**?
- Any **solutions**?

		Player 2		
		rock	scissors	paper
Player 1	rock	0, 0	1, −1	−1, 1
	scissors	−1, 1	0, 0	1, −1
	paper	1, −1	−1, 1	0, 0

Pareto Optimality (1/3)

- We have seen games from the player’s perspective.
- From the point of view of an outside observer, we would like to know if there is some outcome(s) of a game which can be said to be better than others.

Pareto Optimality (2/3)

One outcome o is at least as good for every player as another outcome o', and there is some player who strictly prefers o to o'. In this case, we say o **Pareto-dominates** o'.

Definition

An outcome o^* is Pareto-optimal if there is no other outcome that Pareto-dominates it.

Pareto Optimality (3/3)

		Bob	
		Bach	Stravinsky
Amy	Bach	2, 1	0, 0
	Stravinsky	0, 0	1, 2

		Player 2	
		head	tail
Player 1	head	1, −1	−1, 1
	tail	−1, 1	1, −1

		Criminal *B*	
		Denial	Confess
Criminal *A*	Denial	−1, −1	−9, 0
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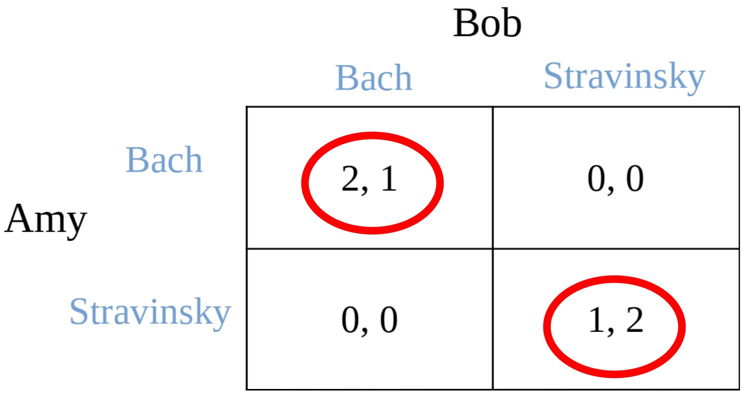
Pareto Optimality (3/3)



		Bob	
		Bach	Stravinsky
Amy	Bach	2, 1	0, 0
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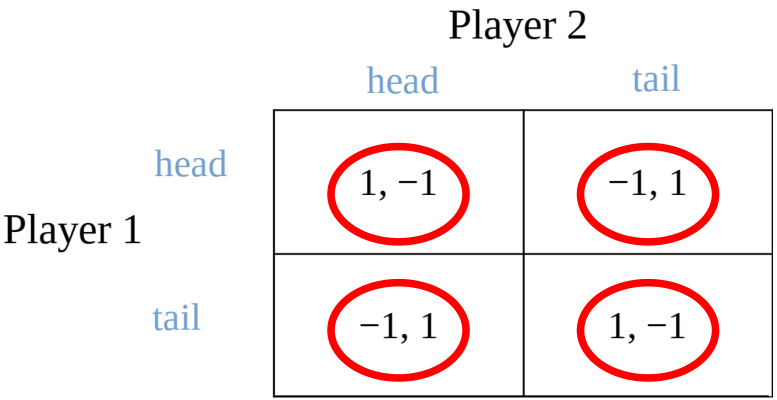
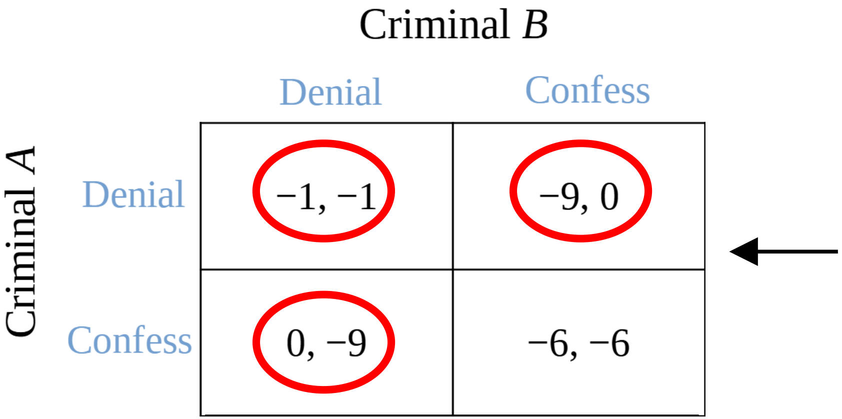


		Player 2	
		head	tail
Player 1	head	1, −1	−1, 1
	tail	−1, 1	1, −1



		Criminal *B*	
		Denial	Confess
Criminal *A*	Denial	−1, −1	−9, 0
	Confess	0, −9	−6, −6

The pure-strategy NE is the only non-Pareto-optimal outcome!

Mixed Strategies

- What we have discussed about are all **pure strategies**.
  - A deterministic action.

Mixed Strategies

- What we have discussed about are all **pure strategies**.
  - A deterministic action.
- What is a **mixed strategy**?

Mixed Strategies

- Like this?
  - Nine-headed Dragon Strike.
- Or like this?
  - Man of many pitches.
- For a portfolio manager in a hedge fund:
  - Portfolio weighting

Back to the Game of Matching Pennies

- Setting the weights?

		Player 2	
		head	tail
Player 1	head	1, −1	−1, 1
	tail	−1, 1	1, −1

Back to the Game of Matching Pennies

- Setting the weights? $0 < \epsilon, \rho < 1$

Player 2

		head	tail	
Player 1	head	1, −1	−1, 1	ρ
	tail	−1, 1	1, −1	$1 - \rho$
		ϵ	$1 - \epsilon$	

Back to the Game of Matching Pennies

- Setting the weights? $0 < \epsilon, \rho < 1$
- The expected utility of player 1 playing ‘head’:

$$f = 1 \cdot \epsilon + (-1) \cdot (1 - \epsilon)$$

- The expected utility of player 1 playing ‘tail’:

$$g = -1 \cdot \epsilon + 1 \cdot (1 - \epsilon)$$

Player 2

		head	tail	
Player 1	head	1, −1	−1, 1	ρ
	tail	−1, 1	1, −1	$1 - \rho$
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An intuitive definition of a **Nash equilibrium**

- A state such that no player can increase her expected payoff (profit, gain, advantage, money, etc.) by a **unilateral** deviation.

- **<u>Nash's Theorem</u>:**

  Every **finite** game (a finite number of players, each has a finite number of pure strategies) has **at least one** Nash equilibrium.

Back to the Game of Matching Pennies

- Setting the weights? $0 < \epsilon, \rho < 1$
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- What if $f \neq g$?

Player 2

Player 1

	head	tail	
head	1, −1	−1, 1	ρ
tail	−1, 1	1, −1	$1 - \rho$
	ϵ	$1 - \epsilon$	

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- What if $f \neq g$?

Consider Player 1's expected utility: $\rho \cdot f + (1 - \rho) \cdot g$

		Player 2		
		head	tail	
Player 1	head	1, −1	−1, 1	ρ
	tail	−1, 1	1, −1	$1 - \rho$
		ϵ	$1 - \epsilon$	

Back to the Game of Matching Pennies

- Setting the weights? $0 < \epsilon, \rho < 1$
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$$g = -1 \cdot \epsilon + 1 \cdot (1 - \epsilon)$$

- **Solving** $f = g \quad \Rightarrow \quad \epsilon = 0.5.$

  Now it’s your turn to solve ρ.

Player 2

		head	tail	
Player 1	head	1, −1	−1, 1	ρ
	tail	−1, 1	1, −1	$1 - \rho$
		ϵ	$1 - \epsilon$	

Back to the Game of Matching Pennies

- Take your time.

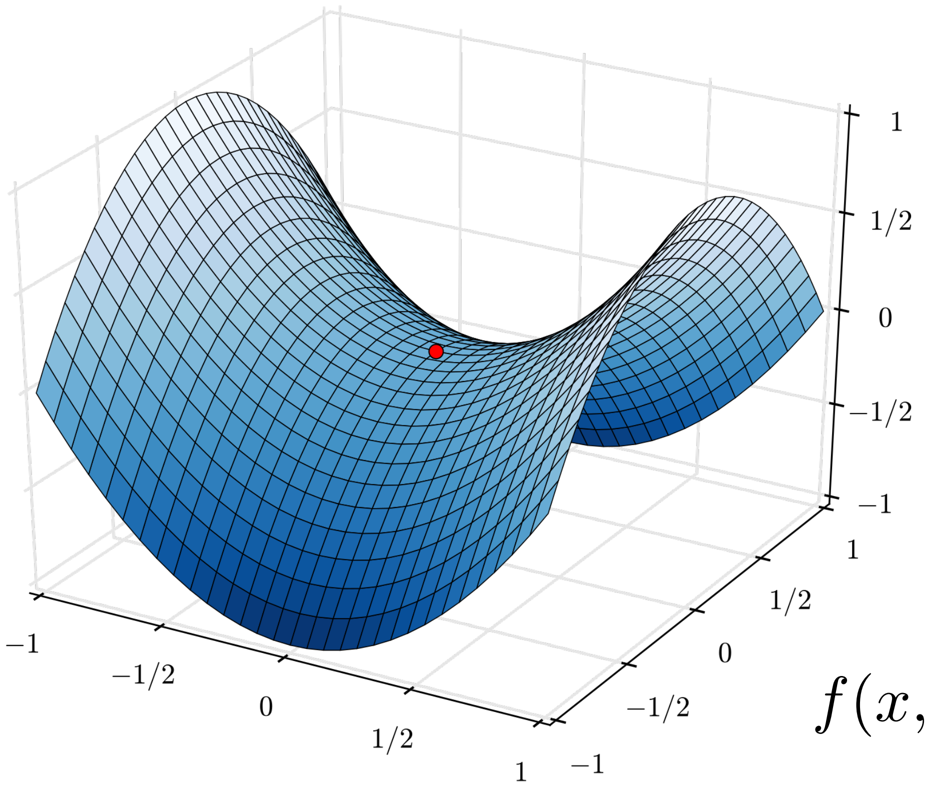
- So we just proved that the game has a kind of solution:

“Mixed-Strategy Nash Equilibrium”.

Player 2

		head	tail	
Player 1	head	1, −1	−1, 1	ρ
	tail	−1, 1	1, −1	$1 - \rho$
		ϵ	$1 - \epsilon$	

Saddle point illustration



$$f(x,y) = x^2 - y^2.$$

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- **Nash's Theorem:**

  Every **finite** game (a finite number of players, each has a finite number of pure strategies) has **at least one** Nash equilibrium.

- The concept of **best responses** & **mixed strategies**.

Back to the classic scene of “A Beautiful Mind”

- https://www.youtube.com/watch?v=2d_dtTZQyUM

- Do you observe anything strange or anything wrong?
  - https://www.youtube.com/watch?v=DTcmmD_MWas

An Easy Exercise

- Please find out a mixed-strategy Nash equilibrium of the rock-scissors-paper game.

Player 2

Player 1

	rock	scissors	paper
rock	0, 0	1, −1	−1, 1
scissors	−1, 1	0, 0	1, −1
paper	1, −1	−1, 1	0, 0