Final Exam of Linear Algebra

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Part I: True (T) or False (F) (30%; each for 3%)

1. For a vector $\mathbf{v} \in \mathbb{R}^n$ and any scalar k, $||k\mathbf{v}|| = k \cdot ||\mathbf{v}||$.
2. Let A be an $n \times n$ matrix, if $A\mathbf{x} = \mathbf{b}$ is consistent for every $n \times 1$ matrix $\mathbf{b}$, then A is invertible.
3. If $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\} \subseteq V$ is a linearly independent set in a vector space V, then $\{\mathbf{u}, \mathbf{v}\}$ is also linearly independent.
4. Given $\mathbf{u} = (2, -1, 3)$, $\mathbf{v} = (-4, 2, -6)$, and $\mathbf{w} = (-3, -2, 1)$, then $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ spans $\mathbb{R}^3$.
5. If A is an $n \times n$ invertible matrix, then $adj(A^\top) = adj(A)^\top$.
6. The basis of a vector space is unique.
7. If {**u**, **v**, **w**} spans a vector space, then {**u**, **v**, **w**, **u**−**v**} also spans the same vector space.
8. $\{x^2 + x, -3x^2 - 2, -5x^2 - 2x\}$ is linearly independent.
9. For any two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, $||\text{proj}_{\mathbf{v}}\mathbf{u}|| \leq ||\mathbf{u}||$.
10. For any two subspaces $S_1, S_2 \subseteq V$ of a vector space V, $S_1 \cup S_2$ is a subspace of V.

Part II: Filling Questions. (80%; each for 5%)

1. Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^4$, $\mathbf{u} = (2, 3, 1, -4)$, $\mathbf{v} = (-1, 2, 0, 1)$, what is $\cos\theta$ for which θ is the angle between $\mathbf{u}$ and $\mathbf{v}$?
2. Given $\mathbf{u} = (3, 0, 2)$, $\mathbf{v} = (0, 1, 1)$, and $\mathbf{w} = (-3, 1, 0)$, represent $(0, 3, 4)$ as a linear combination of $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$. If this is impossible, answer **NO**. ______________________
3. If **a** and **b** are orthogonal vectors, then for every nonzero vector **u**, we have $\text{proj}_{\mathbf{a}}(\text{proj}_{\mathbf{b}}(\mathbf{u})) =$ ____
4. If $A, B \in \mathbb{R}^{3\times 3}$, $\det(A) = 2$, and $\det(B) = 1$, then $\det \begin{pmatrix} 2I_3 & B \\ O & AB^2 \end{pmatrix} =$ _______
5. Let $A = \begin{bmatrix} k-3 & -1 \\ -2 & k-2 \end{bmatrix}$. What's the value(s) of k for which A is invertible? ______
6. Let A be an $n \times n$ square matrix. then if $\det(\lambda A) = k^n \det(A)$, then the value of λ = ______
7. Compute a basis of the null space of A, where $A = \begin{bmatrix} 1 & -1 & 3 \\ 5 & -4 & -4 \\ 7 & -6 & 2 \end{bmatrix}$.
8. Consider the bases $\beta = \{\mathbf{u}_1, \mathbf{u}_2\}$ and $\beta' = \{\mathbf{u}'_1, \mathbf{u}'_2\}$ for $\mathbb{R}^2$, where

   $$\mathbf{u}_1 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 4 \\ -1 \end{bmatrix}, \mathbf{u}'_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \mathbf{u}'_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix},$$

   then compute the transition matrix $[I]^{\beta}_{\beta'}$ from β' to β.
9. Let $S \subseteq V$ be a subspace of a vector space V and $V - \text{span}(S) = \emptyset$. If $\dim(S) = n - 1$, then $\dim(V) =$ ______

10. Given a matrix $A = \begin{bmatrix} 1 & 3 & 1 & 2 & 5 \\ 2 & 6 & 3 & 4 & 10 \\ 1 & 6 & 2 & 2 & 5 \\ -2 & -7 & 5 & -3 & 7 \\ -3 & -9 & 7 & -6 & 10 \end{bmatrix}$. Compute the determinant of A.

11. Compute the distance between the point $(0, 0, 0)$ and the plane $x + 2y - 2z + 9 = 0$.

12. Let $\mathbf{u} = (2, -2, -4), \mathbf{v} = (5, -4, -7), \mathbf{w} = (-3, 1, 0), \mathbf{x} = (-4, 3, h)$, what is the value of h such that $\mathbf{x}$ is in $\mathrm{span}(\{\mathbf{u}, \mathbf{v}, \mathbf{w}\})$?

13. $\begin{vmatrix} ka_{11} & ka_{12} & ka_{13} \\ -k^2a_{21} & -k^2a_{22} & -k^2a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \square \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$. Then, $\square =$ ____.

14. Given $S = \{e^x, e^{x^2}, e^{x^3}\}$, if $W(x)$ is the Wronskian of S, then $W(0) =$ ____.

15. If A is an $n \times n$ invertible matrix, let B be the adjoint matrix of A, then $AB = \square I_n$

16. Compute nullity(A) = $\square$ (i.e., dimension of null(A)), where $A = \begin{bmatrix} 1 & 3 & 1 & 2 \\ 2 & 6 & 3 & 4 \\ 1 & 6 & 2 & 2 \end{bmatrix}$.