Final Exam of Linear Algebra (Makeup)

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Note: Please upload your answer sheet with your signature to iClass before 12:00 24 June 2021.

Part I: True (T) or False (F) (30%; each for 3%)

1. For a vector $\mathbf{v} \in R^n$ and any scalar k, $||k\mathbf{v}|| = k \cdot ||\mathbf{v}||$.
2. For any $n \times n$ matrix *A* and any $n \times n$ matrix *B*, we have $\det(AB) = \det(BA)$.
3. Let *A* be an $n \times n$ matrix, if $A\mathbf{x} = \mathbf{b}$ is consistent for every $n \times 1$ matrix $\mathbf{b}$, then *A* is invertible.
4. Given $\mathbf{u} = (3, 0, 2), \mathbf{v} = (0, 1, -1), \mathbf{w} = (-3, 1, 0)$, then $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are linearly independent.
5. If $S \neq \emptyset$ is a linearly independent set of vectors in a vector space V, then $S \cup \{v\}$ is linear independent for any $v \in V$.
6. Elementary row operations do not change the column space of a matrix.
7. Given $\mathbf{u} = (1, 0, 2), \mathbf{v} = (0, 1, 1)$, $\mathbf{w} = (-3, 1, 0)$, and $\mathbf{x} = (1, -1, 1)$, then $\{\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{x}\}$ is an independent set in R^3.
8. Elementary row operations do not change the rank of a matrix.
9. Given $\mathbf{u} = (2, -1, 3), \mathbf{v} = (-7, -2, 0)$, and $\mathbf{w} = (-6, 3, -9)$, then $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ spans R^3.
10. Elementary row operations may change the dimension of a matrix's column space.

Part II: Single-Answer Multiple Choice. (60%; each for 5%)

1. Given two planes $x + 2y - 2z = 3$ and $2x + 4y - 4z = 5$, the distance between these two planes is :

   (a). 0;

   (b). $\frac{1}{6}$;

   (c). $\frac{7}{6}$;

   (d). 1.

2. Let $\mathbf{u}, \mathbf{v} \in R^4, \mathbf{u} = (2, 3, 1, -4), \mathbf{v} = (-1, 2, 0, 1)$, what is $\cos\theta$ for which θ is the angle between $\mathbf{u}$ and $\mathbf{v}$?

   (a). 1;

   (b). $\sqrt{3}/2$;

   (c). -1;

   (d). 0.

3. According to the orthogonal projection of a vector $\mathbf{v}$ on another vector $\mathbf{w}$, the LENGTH of the vector component of $\mathbf{v}$ projected on $\mathbf{w}$ (that is, along with $\mathbf{w}$) is:

(a). $\frac{\mathbf{w}\cdot\mathbf{v}}{||\mathbf{v}||}$;

(b). $\frac{\mathbf{v}\cdot\mathbf{w}}{||\mathbf{w}||^2}\mathbf{w}$;

(c). $\frac{\mathbf{v}\cdot\mathbf{w}}{||\mathbf{w}||}$;

(d). $\mathbf{v} - \frac{\mathbf{v}\cdot\mathbf{w}}{||\mathbf{w}||^2}\mathbf{w}$.

4. Let $A = \begin{bmatrix} k-3 & -1 \\ -2 & k-2 \end{bmatrix}$. Choose one of the values of k below for which A is invertible.

(a). 4;

(b). -1;

(c). 1;

(d). None of the above.

5. If $A, B \in R^{3\times 3}$, $\det(A) = 2$, and $\det(B) = -1$, then $\det\begin{pmatrix} 2I_3 & B \\ O & AB^2 \end{pmatrix} =$

(a). 16;

(b). -2;

(c). -16;

(d). 1

6. Let A be an $n \times n$ square matrix. then if $\det(R_{1,2}^{(-2)}A) = \lambda \cdot \det(A)$ where $R_{1,2}^{(-2)}$ is an elementary matrix which corresponds to replacing row 2 of A by the sum of row 1 multiplied by -2 and row 2, then λ equals:

(a). -2;

(b). k ;

(c). 1.

(d). -1.

7. Solve the linear system (e.g., using Cramer's rule)

$$\begin{aligned} 4x \quad +5y \quad\quad\quad &= 2 \\ 11x \quad +y \quad +2z &= 3 \\ x \quad +5y \quad +2z &= 1 \end{aligned}$$

(a). $x = \frac{3}{11}, y = \frac{2}{11}, z = -\frac{1}{11}$.

(b). $x = \frac{-3}{11}, y = \frac{-2}{11}, z = \frac{1}{11}$.

(c). $x = \frac{1}{11}, y = \frac{2}{11}, z = -\frac{3}{11}$.

(d). None of the above.

8. If $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ are vectors in R^3, then which one of the following inequalities is a valid Cauchy-Schwarz Inequality (科西不等式)?

(a). $|\mathbf{u} \cdot \mathbf{v}| \geq ||\mathbf{u}|| \, ||\mathbf{v}||$;

(b). $|\mathbf{u} \cdot \mathbf{v}| \leq ||\mathbf{u}|| \, ||\mathbf{v}||$;

(c). $||\mathbf{u} + \mathbf{v}|| \leq ||\mathbf{u}|| + ||\mathbf{v}||$;

(d). None of the above.

9. Consider the bases $B = \{\mathbf{u}_1, \mathbf{u}_2\}$ and $B' = \{\mathbf{u}'_1, \mathbf{u}'_2\}$ for R^2, where

$$\mathbf{u}_1 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 4 \\ -1 \end{bmatrix}, \mathbf{u}'_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \mathbf{u}'_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix},$$

then which one of the following is the transition matrix from B' to B?

(a). $\begin{bmatrix} 0 & -2 \\ -\frac{5}{2} & -\frac{13}{2} \end{bmatrix}$

(b). $\begin{bmatrix} \frac{13}{10} & -\frac{1}{2} \\ -\frac{2}{5} & 0 \end{bmatrix}$

(c). $\begin{bmatrix} \frac{10}{13} & \frac{1}{2} \\ \frac{2}{5} & 0 \end{bmatrix}$

(d). None of the above.

10. If A is an $m \times n$ matrix and $m \geq n$, then which one of the following is CORRECT?

(a). rank(A) $> m$;

(b). rank(A) $> n$;

(c). rank(A) $\leq n$;

(d). rank(A) $= m$.

11. Which one of the following is a basis of the null space of A, where $A = \begin{bmatrix} 1 & -1 & 3 \\ 5 & -4 & -4 \\ 7 & -6 & 2 \end{bmatrix}$

(a). $\left\{ \begin{bmatrix} 13 \\ 7 \\ -1 \end{bmatrix} \right\}$;

(b). $\left\{ \begin{bmatrix} 9 \\ -7 \\ 3 \end{bmatrix} \right\}$;

(c). $\left\{ \begin{bmatrix} 16 \\ 19 \\ 1 \end{bmatrix} \right\}$;

(d). None of the above.

12. Given a matrix $A = \begin{bmatrix} -1 & 2 & 0 & 4 & 5 & -3 \\ 3 & -7 & 2 & 0 & 1 & 4 \\ 2 & -5 & 2 & 4 & 6 & 1 \\ 4 & -9 & 2 & -4 & -4 & 7 \end{bmatrix}$. Which one of the following is WRONG?

(a). rank(A) = 2;

(b). nullity(A) = 4;

(c). dim(row(A)) + dim(col(A)) = 4.

(d). None of the above.

Part III: Fill-In the Boxes (□). (30%; each for 5%)

1. $\begin{vmatrix} ka_{11} & ka_{12} & ka_{13} \\ ka_{21} & ka_{22} & ka_{23} \\ ka_{31} & ka_{32} & ka_{33} \end{vmatrix} = \square \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$.

2. Evaluate $\det(A) = \square$, where $A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 0 & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 0 & 0 & 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 0 & 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$.

3. Given $S = \{\sin 2x, \cos x, x \sin x\}$, if $W(x)$ is the Wronskian of S, then $W(0) = \square$.

4. If A is an $n \times n$ invertible matrix, let B be the adjoint matrix of A, then $\det(AB) = \square$

5. Compute nullity(A) = ☐ (i.e., dimension of null(A)), where $A = \begin{bmatrix} 1 & -1 & 3 \\ 5 & -4 & -4 \\ 7 & -6 & 2 \end{bmatrix}$.

6. For any invertible $n \times n$ matrix A, rank(A) − nullity(A) = ☐.