Preliminaries

Set

- x : element

  x in Set A : $x \in A$.

  x is not in Set A : $x \notin A$.

- $\{1, 2, 3, 4\} = \{1, 3, 2, 4\} = \{1, 4, 2, 3\} = \cdots$

- empty set : $\{\ \}$ or ϕ

- number of elements in A :

  另一種集合表示法 cardinality , $|A|$

  $\Rightarrow A = \{x \mid x \geq 3 \text{ and } x \in \mathbb{Z}\}$ $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$

Subset

- A, B are two sets such that

  for any $\forall x \in A \Rightarrow x \in B$ $\exists$: there exists

  Then A is a subset of B

  That is, $A \subseteq B$ or $B \supseteq A$

  note : $A \nsubseteq B$

  $A \subsetneq B$, $A \subset B$

Operations on Sets

for sets A, B,

- $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$

  "intersection"

- $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$

  "union"

- $A - B = \{x \mid x \in A \text{ and } x \notin B\}$

  "difference"

- $\overline{A} = \{x \mid x \notin A\}$

  "complement"

Example:

Try to test these in Python

Suppose $U = \{1, 2, \dots, 9, 10\}$.

$A = \{1, 2, 3, 4\} \subseteq U$.

$B = \{3, 4, 5, 6\} \subseteq U$

Then

(1) $A \cap B = \{3, 4\}$

(2) $A \cup B = \{1, 2, 3, 4, 5, 6\}$

(3) $A - B = \{1, 2\}$

(4) $\overline{A} = \{5, 6, 7, 8, 9, 10\}$

Suppose $A_1, A_2, \cdots, A_n$ are n sets.

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_n$$

$$= \{ x \mid x \in A_i \text{, for all } \underbrace{i = 1, 2, \cdots, n}_{i \in [n]} \}$$

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \cdots \cup A_n$$

$$= \{ x \mid x \in A_i \text{, for some } i = 1, 2, \cdots, n \}$$

Cartesian product (卡氏積)

Sets A, B

$$A \times B = \{ \underbrace{(a, b)}_{\text{ordered pair (有序數對)}} \mid a \in A,\ b \in B \}$$

Function

A, B : sets

$f : A \longmapsto B$

$x \longmapsto f(x)$ unique

A : domain of f 定義域

B : codomain of f 對應域

if $S \subseteq A$, $f(S) = \{f(x) \mid x \in S\}$

if $S' \subseteq B$, $f^{-1}(S') = \{x \in A \mid f(x) \in S'\}$

* $f(A) = \{f(x) \mid x \in A\}$: range of f (值域)

Example :

$f : [-5, 5] \to \mathbb{R}$, $f(x) = x^2$

range of f : $[0, 25]$

if $S = [1, 3]$, then $f(S) = [1, 9]$

if $S' = [4, 9]$, then $f^{-1}(S') = [-3, -2] \cup [2, 3]$

1對1 映成

One-to-One / Onto

Suppose $f: A \to B$ is a function

(1) if $f(x) = f(y) \Rightarrow x = y$

then f is "one-to-one"

(2) if $f(A) = B$,

then f is "onto"

$\Updownarrow$

$\forall y \in B, \exists x \in A$, s.t. $f(x) = y$

note

One-to-one : injection

Onto : surjection

$\left.\begin{array}{l}\text{One-to-one : injection} \\ \text{Onto : surjection}\end{array}\right\} \xrightarrow{\text{and}} \text{bijection}$

Example :

$f: [-1, 1] \longmapsto [0, 1]$, $f(x) = x^2$

Since $f(-1) = f(1) = 1$, f is NOT one-to-one

Composition 函數合成

Suppose $f: A \mapsto B$, $g: B \mapsto C$ are two functions

Then $g \circ f : A \mapsto C$ is defined as

$$(g \circ f)(x) = g(f(x))$$

$$x \to \underset{\text{machine}}{\boxed{f}} \to f(x) \to \underset{\text{machine}}{\boxed{g}} \to g(f(x))$$

Example: $f(x) = x^2$, $g(x) = e^x$

$f: \mathbb{R} \mapsto \mathbb{R}$, $g: \mathbb{R} \mapsto \mathbb{R}$

$$(g \circ f)(x) = g(f(x)) = g(x^2) = e^{x^2}$$

$$(f \circ g)(x) = f(g(x)) = f(e^x) = e^{2x}$$

Invertible function 可逆函數

Suppose $f: A \mapsto B$ is a fuction

if $\exists\ g: B \mapsto A$ such that

$$g \circ f = I_A \text{ and } f \circ g = I_B$$

Then f is invertible

$I_A : A \mapsto A$ such that $I_A(x) = x$, $\forall x \in A$

f is an invertible function

$\Leftrightarrow$ f is one-to-one and onto

note: $(f^{-1})^{-1} = f$, f^{-1} is also invertible

Example

$f: \mathbb{R} \mapsto \mathbb{R}$, $f(x) = 2x+3$,

$\Rightarrow$ f is one-to-one and onto. (Check!)

$f^{-1}: \mathbb{R} \mapsto \mathbb{R}$,

$$f^{-1}(x) = \frac{x-3}{2}$$

Let $2x+3 = y$

$\Rightarrow x = \frac{y-3}{2}$

Field 體 examples: $\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{C}$, $\mathbb{C}^2$, ...

Suppose $\mathbb{F}$ is a set, and we define/have addition $(+)$ and multiplication $(\cdot)$ operations on $\mathbb{F}$,

such that

(1) $\forall a, b \in \mathbb{F}$, $a+b \in \mathbb{F}$ and $a \cdot b \in \mathbb{F}$ 加/乘法封閉性

(2) $\forall a, b \in \mathbb{F}$, $a+b = b+a$ and $a \cdot b = b \cdot a$ 加/乘法交換性

(3) $\forall a, b, c \in \mathbb{F}$, $(a+b)+c = a+(b+c)$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ 加/乘法結合律

(4) $\exists\, 0, 1 \in \mathbb{F}$ such that $\forall a \in \mathbb{F}$, $0+a = a$ and $1 \cdot a = a$ 加/乘法單位元素

(5) $\forall a \in \mathbb{F}$, $\exists\, b \in \mathbb{F}$ such that $a+b=0$ 加法反元素

(6) $\forall a \in \mathbb{F}$, $a \neq 0$, $\exists\, b \in \mathbb{F}$ such that $a \cdot b = 1$ 乘法反元素

(7) $\forall a, b, c \in \mathbb{F}$, $a \cdot (b+c) = a \cdot b + a \cdot c$ 乘法對加法分配律

Then we say $\mathbb{F}$ is a field.

Example $\mathbb{Z}_2 = \{0, 1\}$

$+$	0	1
0	0	1
1	1	0

$\cdot$	0	1
0	0	0
1	0	1

Recall:

$\mathbb{F}^2$: $\mathbb{R}^2$, $\mathbb{C}^2$

$\mathbb{F}^n$: $\mathbb{R}^n$, $\mathbb{C}^n$

Recall :

$\mathbb{F}^2$: $\mathbb{R}^2, \langle +_2, \cdot_2 \rangle$

$\mathbb{R}^n$: $\mathbb{R}^n, \langle +_n, \cdot_n \rangle$