

Preliminaries

Set

- x : element of A : $x \in A$.
- x is not in Set A : $x \notin A$.
- $\{1, 2, 3, 4\} = \{1, 3, 2, 4\} = \{1, 4, 2, 3\} = \dots$
- empty set : $\{\}$ or \emptyset
- number of elements in A :

另一種集合表示法 cardinality, $|A|$
 $\Rightarrow A = \{x \mid x \geq 3 \text{ and } x \in \mathbb{Z}\}$ N, Z, Q, R, C

Subset

- A, B are two sets such that

for any $\forall x \in A \Rightarrow x \in B$ \exists : there exists

Then A is a subset of B

That is, $A \subseteq B$ or $B \supseteq A$

Note : $A \not\subseteq B$

$A \subsetneq B$, $A \subset B$

$\{1, 2, 3, 4, 5, 6\} = B \cup A$

$\{5, 6\} = B - A$

$\{1, 2, 3, 4, 5, 6\} = \bar{B}$

Operations on Sets

for sets A, B ,

- $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$ "intersection" $\rightarrow x \in A \cap B \iff x \in A \text{ and } x \in B$
- $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$ "union" $\rightarrow \{1, 2, 3\} \cup \{3, 4, 5\} = \{1, 2, 3, 4, 5\}$
- $A - B = \{x \mid x \in A \text{ and } x \notin B\}$ "difference" $\rightarrow \{1, 2, 3\} - \{3, 4, 5\} = \{1, 2\}$
- $\bar{A} = \{x \mid x \notin A\}$ "complement" $\rightarrow \bar{\{1, 2, 3\}} = \{4, 5, 6, 7, 8, 9, 10\}$

Example:

Try to test these
in Python

Suppose $U = \{1, 2, \dots, 9, 10\}$. $\rightarrow x \in A \iff x \in U \setminus A$

$$A = \{1, 2, 3, 4\} \subseteq U.$$

$$B = \{3, 4, 5, 6\} \subseteq U$$

Then

$$(1) A \cap B = \{3, 4\}$$

$$(2) A \cup B = \{1, 2, 3, 4, 5, 6\}$$

$$(3) A - B = \{1, 2\}$$

$$(4) \bar{A} = \{5, 6, 7, 8, 9, 10\}$$

Suppose A_1, A_2, \dots, A_n are n sets.

$$\bigcap_{i=1}^n A_i = A_1 \cap A_2 \cap A_3 \cap \dots \cap A_n$$
$$= \{x \mid x \in A_i, \text{ for all } i = 1, 2, \dots, n\}$$

\downarrow
 $i \in [n]$

$$\bigcup_{i=1}^n A_i = A_1 \cup A_2 \cup \dots \cup A_n$$
$$= \{x \mid x \in A_i, \text{ for some } i = 1, 2, \dots, n\}$$

Cartesian product (笛氏積)

Sets A, B

$$A \times B = \{(a, b) \mid a \in A, b \in B\}$$

↳ ordered pair (有序數對)

Since $f(-1) = f(1) = \{x \in f_0\}$, not the pair

$$[f, 1] = \{2\} \text{ and } [f, -1] = \{2\}$$

$$[f, 1] \cup [f, -1] = \{2\} \text{ and } [f, 0] = \{2\}$$

Function

A, B : sets

$f: A \rightarrow B$

$x \mapsto f(x)$

unique

A : domain of f 是義域

B : codomain of f 對應二域

if $S \subseteq A$, $f(S) = \{f(x) | x \in S\}$

if $S' \subseteq B$, $f^{-1}(S') = \{x \in A | f(x) \in S'\}$

* $f(A) = \{f(x) | x \in A\}$: range of f (值域)

Example

$f: [-5, 5] \rightarrow \mathbb{R}$, $f(x) = x^2$

range of f : $[0, 25]$

if $S = [1, 3]$, then $f(S) = [1, 9]$

if $S' = [4, 9]$, then $f^{-1}(S') = [-3, -2] \cup [2, 3]$

$$\Rightarrow A \cap B = \{3, 4, 5\}$$

$$\Rightarrow A \cup B = f([-5, 5] \cup [4, 9])$$

$$\Rightarrow A - B = \{1, 2\}$$

One-to-One / Onto

Suppose $f: A \rightarrow B$ is a function

(1) if $f(x) = f(y) \Rightarrow x = y$

then f is "one-to-one"

(2) if $f(A) = B$,

\Updownarrow then f is "onto"

$\forall y \in B, \exists x \in A, \text{ s.t. } f(x) = y$

Note:

One-to-one : injection

Onto : surjection

$\xrightarrow{\text{and}}$ bijection

Example:

$f: [-1, 1] \rightarrow [0, 1], f(x) = x^2$

Since $f(-1) = f(1) = 1$, f is NOT one-to-one

$$I = f \circ f^{-1}, I = f \circ g$$

additional info

$A = X^A, \forall n \in \mathbb{N}$ function $A \rightarrow A : \mathbb{N}$

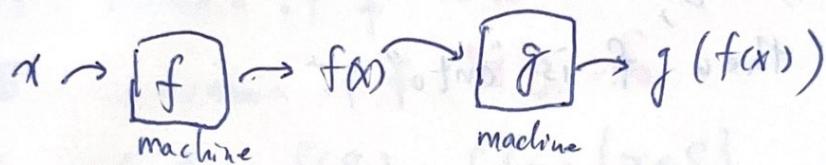
Composition

函数合成

Suppose $f: A \rightarrow B$, $g: B \rightarrow C$ are two functions

Then $g \circ f: A \rightarrow C$ is defined as

$$(g \circ f)(x) = g(f(x))$$



Example : $f(x) = x^2$, $g(x) = e^x$

$$f: \mathbb{R} \rightarrow \mathbb{R}, g: \mathbb{R} \rightarrow \mathbb{R}$$

$$(g \circ f)(x) = g(f(x)) = g(x^2) = e^{x^2}$$

$$(f \circ g)(x) = f(g(x)) = f(e^x) = e^{2x}$$

Invertible function 可逆函数

Suppose $f: A \rightarrow B$ is a function

if $\exists g: B \rightarrow A$ such that

$$g \circ f = I_A \text{ and } f \circ g = I_B$$

Then f is invertible

$I_A: A \rightarrow A$ such that $I_A(x) = x, \forall x \in A$

f is an invertible function $\Rightarrow f^{-1}$ is also invertible

$\Leftrightarrow f$ is one-to-one and onto

Note: $(f^{-1})^{-1} = f$, f^{-1} is also invertible

Example

$$f: \mathbb{R} \rightarrow \mathbb{R}, f(x) = 2x + 3$$

$\Rightarrow f$ is one-to-one and onto (check)

$$f^{-1}: \mathbb{R} \rightarrow \mathbb{R}, f^{-1}(x) = \frac{x-3}{2}$$

$$\text{Let } 2x+3 = y \\ \Rightarrow x = \frac{y-3}{2}$$

$$(2x+3)+a = 2x+a$$

$$(2x+3) \cdot a = 2x \cdot a$$

$$a = d + m$$

$$d = a \cdot b$$

$$m = a \cdot c$$

$$a = d + m$$

$$d = a \cdot b$$

$$c \cdot a + d \cdot a = (c+d) \cdot a$$

$$a = d + m$$

$$1.07 = 1 + 0.07$$

1	0	.	7	0	0	0	0
1	0	.	7	0	0	0	0

Field 領域

examples: $\mathbb{R}, \mathbb{R}^2, \mathbb{C}, \mathbb{C}^2, \dots$

Suppose F is a set, and we define / have addition and multiplication operations on F .

such that

(1) $\forall a, b \in F, a+b \in F$ 加/乘法封閉性
and $a \cdot b \in F$

(2) $\forall a, b \in F, a+b = b+a$ and $a \cdot b = b \cdot a$ 加/乘法交換性

(3) $\forall a, b, c \in F, (a+b)+c = a+(b+c)$
and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ 加/乘法結合律

(4) $\exists 0, 1 \in F$ such that

$\forall a \in F, 0+a = a$ 加/乘法單位元素
and $1 \cdot a = a$

(5) $\forall a \in F, \exists b \in F$ such that $a+b=0$ 加法反元

(6) $\forall a \in F, a \neq 0, \exists b \in F$ such that $a \cdot b = 1$ 乘法反元

(7) $\forall a, b, c \in F, a \cdot (b+c) = a \cdot b + a \cdot c$
乘法對加法分配律

Then we say F is a field.

Example $\mathbb{Z}_2 = \{0, 1\}$

+	0	1
0	0	1
1	1	0

Recall:

$\mathbb{F}^2 := \mathbb{R}^2, \subset$

$\mathbb{R}^n := \mathbb{R}^n, \subset$

Recall:

Variation set of all materials form initial point until A

$\mathbb{F}^2 = \mathbb{R}^2, \langle t_1, \cdot \rangle$ in general not to return to t_1

$\mathbb{R}^n = \mathbb{R}^n, \langle t_n, \cdot \rangle$ not $t_1 + \dots + t_n$

$\text{span}(v_1, \dots, v_n) = \{a_1v_1 + \dots + a_nv_n \mid a_i \in \mathbb{F}\}$

closed

If $\text{span}(x, y, z) = \{x + (y - x)t_1 + (z - x)t_2 \mid t_1, t_2 \in \mathbb{F}\}$

then x is spans V

JUA

Example variation may need a TCM in $(2x - f)$

~~Suppose $\{x, y, z\} \subset V$ then x, y, z are linearly independent~~

This implies $(x, y, z) \text{ is a basis of } V$

spans V

$$\begin{cases} xP + yQ = f \\ xDx - yA = g \end{cases}$$

(Solve for)

$$\begin{cases} xP + yQ = f \\ xDx - yA = g \end{cases}$$

Copper (Monoxide) $xP + yQ = f$

$$x(1, 0, 0, 0) + y(0, 1, 0, 0) + 0(0, 0, 1, 0) + 0(0, 0, 0, 1) = f$$

$$x(1, 0, 0, 0) + y(0, 1, 0, 0) + 0(0, 0, 1, 0) + 0(0, 0, 0, 1) = f$$

$$(1, 0, 0, 0) + y(0, 1, 0, 0) + 0(0, 0, 1, 0) + 0(0, 0, 0, 1) = f$$