§ Systems of Linear Equations and Matrices

Matrices often appear as tables of numerical data that arise from physical observations. / mathematical contexts.

For example,

$$5x + y = 3$$
$$2x - y = 4$$

$$\Rightarrow \begin{bmatrix} 5 & 1 & 3 \\ 2 & -1 & 4 \end{bmatrix}$$

◎ We will show that the solution of the above system can be obtained by performing appropriate operations on the matrix.

◎ This is particularly important in developing computer programs for solving systems of equations.

※ Matrices can be also viewed as mathematical objects in their own right.

$\Rightarrow$ The study of matrices and related topics : "Linear Algebra"

<u>Linear Equation</u> :

Given n variables $x_1, x_2, \cdots, x_n$ and constants $a_1, a_2, \cdots, a_n$ and b (over a field), a linear equation is the one that can be expressed in the form :

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b \quad \cdots\cdots (1)$$

Note that the a's are NOT all zero.

△ When $b=0$, equation (1) is called homogeneous.

Example :

$$\begin{cases} x+3y=7, & x_1-2x_2-3x_3+x_4=0, \\ x_1+x_2+\cdots+x_n=1 \\ \frac{1}{2}x-y+3z=-1 \end{cases}$$

→ linear equations

But

$x+3y^2=4$, $3x+2y-xy=5$

$\sin x+y=0$, $\sqrt{x_1}+2x_2+x_3=1$

are NOT linear equations.

△ A finite set of linear equations

⇒ a system of linear equations.

A general linear system of m equations in the n unknowns (variables) $x_1, x_2, \cdots, x_n$ can be written as

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad\qquad \vdots \qquad\qquad \vdots \quad &\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

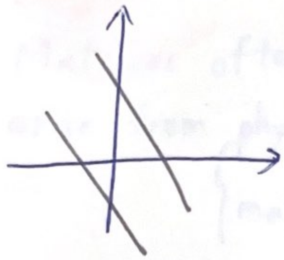
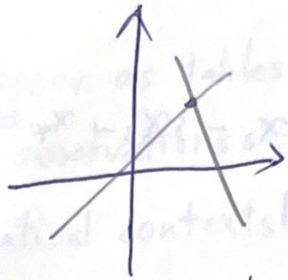
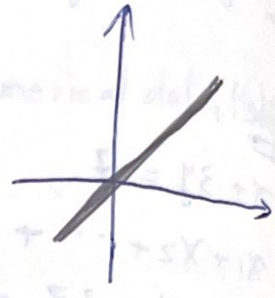
A solution of above system of linear equations :

$x_1=s_1$, $x_2=s_2$, $\cdots$, $x_n=s_n$

can be written as $(s_1, s_2, \cdots, s_n)$

↳ ordered n-tuple.

Linear systems in Two unknows :

no solution   One solution   Infinitely many solutions

For linear systems in Three unknowns,

we have "no solutions", "One solution", or "infinitely many solutions", too!

<u>Example</u> :

Solve
$$\begin{aligned} x - y &= 1 \\ 2x + y &= 6 \end{aligned}$$

$$\Rightarrow \quad \begin{aligned} x - y &= 1 \\ 3y &= 4 \end{aligned}$$

$\Rightarrow y = \frac{4}{3}, \ x = 1 + y = \frac{7}{3}$

$\therefore$ We have exactly one solution.

<u>Example</u> :
$$\left.\begin{aligned} x + y &= 4 \\ 3x + 3y &= 6 \end{aligned}\right] \times(3)$$

$$\Rightarrow \quad \begin{aligned} -3x + (-3y) &= -12 \\ 3x + 3y &= 6 \end{aligned}$$

$\Rightarrow 0 = -6 \quad (\Rightarrow\Leftarrow)$

$\therefore$ No solution

Example : Solve the linear system :

$$4x - 2y = 1$$
$$16x - 8y = 4$$
$\times(-4)$

$$4x - 2y = 1$$
$$\Rightarrow \quad 0 = 0 \rightarrow$$ Always true and can be omitted

$$\Rightarrow 4x - 2y = 1$$
$$x = \frac{1}{4} + \frac{1}{2}y$$

Then we assign an arbitrary value (i.e, parameter) t to y, and we have $\begin{cases} x = \frac{1}{4} + \frac{1}{2}t \\ y = t \end{cases}, \ t \in \mathbb{R}$

We have infinitely many solutions because we have infinitely many values of t.

* augmented matrix :

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\vdots \qquad \vdots \qquad \qquad \vdots \qquad \vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

We can abbreviate the system by

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix} \rightarrow$$ augmented matrix

Later in the future, we will show that solving the linear system corresponds to elementary row operations.

Elementary Row Operations

row $\leftrightarrow$ equation

1. Multiply a $\boxed{\text{row}}$ through by a nonzero constant.
2. Interchange two $\boxed{\text{rows}}$
3. Add a constant times one $\boxed{\text{row}}$ to another

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{bmatrix} \times(-2)+ \Rightarrow \begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 3 & 6 & -5 & 0 \end{bmatrix} \times(-3)+$$

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 3 & -11 & -27 \end{bmatrix} \times\left(\tfrac{1}{2}\right) \Rightarrow \begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 3 & -11 & -27 \end{bmatrix} \times(-3)+$$

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & -\frac{1}{2} & -\frac{3}{2} \end{bmatrix} \times(-2) \Rightarrow \begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & 1 & 3 \end{bmatrix} (-1)$$

$$\begin{bmatrix} 1 & 0 & \frac{11}{2} & \frac{35}{2} \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & 1 & 3 \end{bmatrix} \times\left(-\tfrac{11}{2}\right)+ \quad \times\left(\tfrac{7}{2}\right)+ \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

reduced row echelon form (RREF)

!!

$x_1 + 0 + 0 = 1$

$0 + x_2 + 0 = 2$

$0 + 0 + x_3 = 3$

Reduced Row Echelon Form

1. "Leading 1": For each row, the first nonzero number is 1.

2. Group together the all-zero rows.

3. For any two successive rows that do not consist all-zero rows, the leading 1 in the lower row occurs "farther" to the right than that in the higher row.

   (i.e., $\begin{matrix}1&&&&\\&1&&&\\&&1&&\\&&&\ddots&\\&&&&1\end{matrix}$)

4. Each column that contains a leading 1 has zeros everywhere else in that column.

Examples:

$$\begin{bmatrix}1&0&0&4\\0&1&0&7\\0&0&1&-1\end{bmatrix},\quad \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}.$$

$$\begin{bmatrix}0&1&-2&0&1\\0&0&0&1&3\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix},\quad \begin{bmatrix}0&0\\0&0\end{bmatrix}.$$

Note

$$\begin{cases} x-y+2z=5 \\ 2x-2y+4z=10 \\ 3x-3y+6z=15 \end{cases} \Rightarrow \begin{bmatrix} 1 & -1 & 2 & 5 \\ 2 & -2 & 4 & 10 \\ 3 & -3 & 6 & 15 \end{bmatrix} \begin{array}{l} \times(-2) \\ \times(-3) \end{array}$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 2 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Updownarrow$$

$$x-y+2z=5$$

Let $y=r$, $z=s$, $r,s\in\mathbb{R}$.

Then $\begin{cases} x = 5+r-2s \\ y=r \\ z=s \end{cases}$, for $r,s\in\mathbb{R}$

Gauss-Jordan elimination

while the entire matrix is NOT in row echelon form (REF)

1. 找出最左邊不全為0的 column C
2. 利用row-interchange使C中最上方的entry不為0
3. 利用scalar multiplication使C中最上方的entry為1
4. 利用乘上一個scalar於最上列至下方各列，使最上列的 leading 1 下方所有entries為0
5. 將矩陣最上列註記消掉

6. 得到一個 REF，然後將它轉為RREF (利用 $\times(\lambda)$)

Example:

$$x_1 + 3x_2 - 2x_3 + 2x_5 = 0$$

$$2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 = -1$$

$$5x_3 + 10x_4 + 15x_6 = 5$$

$$2x_1 + 6x_2 + 8x_4 + 4x_5 + 18x_6 = 6$$

(Sol):

The augmented matrix for the system:

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 2 & 6 & 0 & 8 & 4 & 18 & 6 \end{bmatrix} \begin{array}{l} \times(-2) \quad \times(-2) \\ \\ \\ \\ \end{array}$$

$$\Rightarrow \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{bmatrix} \begin{array}{l} \\ \times(-1) \\ \\ \\ \end{array}$$

$$\Rightarrow \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{bmatrix} \begin{array}{l} \\ \times(-5) \quad \times(-4) \\ \\ \\ \end{array}$$

$$\Rightarrow \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \end{bmatrix} \begin{array}{l} \\ \\ \text{interchange} \\ \\ \end{array}$$

$$\Rightarrow \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & \cancel{6}\,1 & \cancel{2}\,\frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \times \left(\frac{1}{6}\right) \rightarrow \text{a REF}$$

$$\Downarrow \text{RREF}$$

$$\begin{bmatrix} 1 & 3 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{RREF}$$

$$\therefore \begin{cases} x_1 + 3x_2 + 4x_4 + 2x_5 = 0 \\ x_3 + 2x_4 = 0 \\ x_6 = \frac{1}{3} \end{cases}$$

$$\Rightarrow \begin{cases} x_1 = -3x_2 - 4x_4 - 2x_5 \\ x_3 = -2x_4 \\ x_6 = \frac{1}{3} \end{cases}$$

Let $x_2 = r$, $x_4 = s$, $x_5 = t$, for $r, s, t \in \mathbb{R}$

$$\Rightarrow \begin{cases} x_1 = -3r - 4s - 2t, \\ x_2 = r, \\ x_3 = -2s, \\ x_4 = s, \\ x_5 = t, \\ x_6 = \frac{1}{3} \end{cases}, \quad r, s, t \in \mathbb{R}$$

<u>Note</u>: (1) Every matrix has a unique RREF

(2) The REFs of a matrix are NOT unique.

A linear system is **consistent** if it has ≥ 1 solution.

Example :

$$\begin{cases} x-y=1 \\ 2x+y=6 \end{cases}$$

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & 6 \end{bmatrix} \xrightarrow{\times(-2)} \Rightarrow \begin{bmatrix} 1 & -1 & 1 \\ 0 & 3 & 4 \end{bmatrix} \times \frac{1}{3}$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 4/3 \end{bmatrix} \times 1$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 7/3 \\ 0 & 1 & 4/3 \end{bmatrix}$$

$\therefore$ $x = \frac{7}{3}$, $y = \frac{4}{3}$

But

$$\begin{cases} x+y=4 \\ 3x+3y=6 \end{cases} \quad \begin{bmatrix} 1 & 1 & 4 \\ 3 & 3 & 6 \end{bmatrix} \times \frac{1}{3}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 4 \\ 1 & 1 & 2 \end{bmatrix} \times(-1)$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 4 \\ 0 & 0 & -2 \end{bmatrix}$$

$$0x + 0y = -2$$

No solution, inconsistent

$$\begin{cases} 4x-2y=1 \\ 16x-8y=4 \end{cases} \Rightarrow \begin{bmatrix} 4 & -2 & 1 \\ 16 & -8 & 4 \end{bmatrix} \begin{matrix} \times\frac{1}{4} \\ \times\frac{1}{4} \end{matrix}$$

$$\Rightarrow \begin{bmatrix} 1 & -\frac{1}{2} & \frac{1}{4} \\ 4 & -2 & 1 \end{bmatrix} \times(-4)$$

$$\Rightarrow \begin{bmatrix} 1 & -\frac{1}{2} & \frac{1}{4} \\ 0 & 0 & 0 \end{bmatrix}$$

$$x-\frac{1}{2}y=\frac{1}{4}$$

Let $y=t$, $\Rightarrow x=\frac{1}{4}+\frac{1}{2}t$ $\therefore \begin{cases} x=\frac{1}{4}+\frac{1}{2}t \\ y=t \end{cases}$, $t\in\mathbb{R}$

Homogeneous Linear systems

A system of linear equations with all the constant terms being zero.

That is,

$$\begin{aligned} a_{11}x_1+a_{12}x_2+\dots+a_{1n}x_n&=0 \\ a_{21}x_1+a_{22}x_2+\dots+a_{2n}x_n&=0 \\ &\vdots \\ a_{m1}x_1+a_{m2}x_2+\dots+a_{mn}x_n&=0 \end{aligned}$$

Trivial solution: $x_1=x_2=\dots=x_n=0$

If there are other solutions, they are called "nontrivial solutions"

Homogeneous system of linear equations (homogeneous linear system)
: only two possibilities for its solutions:

$$\begin{cases} \text{① only the trivial solution} \\ \text{② infinitely many solutions} \end{cases}$$

Solving By Back_Substitution

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \times 2$$

$$\Rightarrow \begin{bmatrix} 1 & 3 & 0 & 4 & 2 & 6 & 2 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \times(-6) \quad (-3)$$

$$\Rightarrow \begin{bmatrix} 1 & 3 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore x_6 = \frac{1}{3}$

Let $x_5 = t$

$x_3 + 2x_4 = 0$

Let $x_4 = s, \Rightarrow x_3 = -2s$

Let $x_2 = r$

$x_1 + 3x_2 + 4x_4 + 2x_5 = 0$

$\Rightarrow x_1 + 3r + 4s + 2t = 0$

$\Rightarrow x_1 = -3r - 4s - 2t$

☆ the RREF of the augmented matrix has r nonzero rows

$\Rightarrow$ $n-r$ free variables

Step 1:

$x_1 = -3x_2 + 2x_3 - 2x_5$

$x_3 = 1 - 2x_4 - 3x_6$

$x_6 = \frac{1}{3}$

Step 2 ①

$x_1 = -3x_2 + 2x_3 - 2x_5$

$x_3 = -2x_4$

$x_6 = \frac{1}{3}$

Step 2 ②

$x_1 = -3x_2 - 4x_4 - 2x_5$

$x_3 = -2x_4$

$x_6 = \frac{1}{3}$

Step 3: $x_1 = -3r - 4s - 2t$,

$x_2 = r$, $\quad x_6 = \frac{1}{3}$

$x_3 = -2s$,

$x_4 = s$,

$x_5 = t$,