Matrix & Matrix Operations

Matrix: rectangular array of numbers

entries: the numbers in the array

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

$$A = [a_{ij}]_{m \times n} \quad \text{or} \quad A = [a_{ij}]$$

$$(A)_{ij} = a_{ij}$$

$\vec{a} = [a_1 \ a_2 \cdots a_n] \rightarrow$ a $1 \times n$ row vector

$b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \rightarrow$ a $m \times 1$ column vector

* $m = n$: square matrix

$$\begin{bmatrix} a_{11} & & & \\ & a_{22} & & \\ & & \ddots & \\ & & & a_{nn} \end{bmatrix} \rightarrow \text{main diagonal of } A$$

* equal

For an $m_1 \times n_1$ matrix A & an $m_2 \times n_2$ matrix B, A and B are **equal**.

if $m_1 = m_2$, $n_1 = n_2$ and $(A)_{ij} = (B)_{ij}$ for $i \in [m_1]$, $j \in [n_1]$

corresponding entries

Summation

If $A = [a_{ij}]$, $B = [b_{ij}]$ have the same size,

then $(A \pm B)_{ij} = (A)_{ij} \pm (B)_{ij} = a_{ij} \pm b_{ij}$

Example:

$$A = \begin{bmatrix} 2 & 1 & 0 & 3 \\ -1 & 0 & 2 & 4 \\ 4 & -2 & 7 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$$

$A + C$?? $A - C$??

Scalar multiple

For any matrix A $\left(= [a_{ij}]\right)$ and scalar c,

$$(cA)_{ij} = c(A_{ij}) = ca_{ij}$$

Product: For two matrices $A: m \times r$, $B: r \times n$

The product, denoted by AB, is the $m \times n$ matrix whose entries are determined as

$$(AB)_{ij} = a_{i1} \cdot b_{1j} + a_{i2} \cdot b_{2j} + a_{i3} \cdot b_{3j} + \dots + a_{ir} \cdot b_{rj}$$
$$= \sum_{k=1}^{r} a_{ir} \cdot b_{rj}$$

"A Matrix Can be Partitioned"

Example :

$$A = \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ \hline a_{31} & a_{32} & a_{33} & a_{34} \end{array}\right] = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

a submatrix

or $$A = \left[\begin{array}{cccc} a_{11} & a_{12} & a_{13} & a_{14} \\ \hline a_{21} & a_{22} & a_{23} & a_{24} \\ \hline a_{31} & a_{32} & a_{33} & a_{34} \end{array}\right] = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}$$

row vectors

or $$A = \left[\begin{array}{c|c|c|c} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{array}\right] = \begin{bmatrix} c_1 & c_2 & c_3 & c_4 \end{bmatrix}$$

column vectors

So,

$$AB = A[b_1\ b_2 \cdots b_n] = [Ab_1\ Ab_2 \cdots Ab_n]$$

AB computed "column-by-column"

$$AB = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix} \cdot B = \begin{bmatrix} a_1 B \\ a_2 B \\ \vdots \\ a_m B \end{bmatrix}$$

AB computed row-by-row

Example :

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix} = [12\ \ 27\ \ 30\ \ 13]$$

or $$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} B = \begin{bmatrix} [1\ 2\ 4]\, B \\ [2\ 6\ 0]\, B \end{bmatrix} \begin{bmatrix} 27 \\ -4 \end{bmatrix}$$

GO TO 3/9 END

Matrix products as Linear Combinations

$A_1, A_2, \cdots, A_r$: matrices of the same size

$c_1, c_2, \cdots, c_r$: scalars

then

$c_1A_1 + c_2A_2 + \cdots + c_rA_r$ is a linear combination of $A_1, \cdots A_r$ with coefficients $c_1, c_2, \cdots, c_r$

Matrix Product $\Rightarrow$ linear combinations?

$$A = \begin{array}{c} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix}_{1\times n} \\ \quad c_1 \qquad c_2 \qquad\quad c_n \end{array}, \qquad \mathbf{x} = \begin{bmatrix} \boxed{x_1} \\ \boxed{x_2} \\ \vdots \\ \boxed{x_n} \end{bmatrix}_{n\times 1}$$

$c_1x_1 + c_2x_2 + \cdots + c_nx_n$

Then

$$A\mathbf{x} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix}$$

$$= x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

Theorem

If A is an $m\times n$ matrix and $\mathbf{x}$ is an $n\times 1$ column vector then $A\mathbf{x}$ can be expressed as a linear combination of the column vectors of A in which the coefficients are the entries of $\mathbf{x}$.

Example :

$$\begin{bmatrix} -1 & 3 & 2 \\ 1 & 2 & -3 \\ 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ -3 \end{bmatrix}$$

$$= (2)\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} - 1\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix}$$

$$\left(\begin{bmatrix} 1 \end{bmatrix} \begin{bmatrix} 2 \end{bmatrix} \begin{bmatrix} 3 \end{bmatrix} \right) \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix} 1 + \begin{bmatrix} 2 \end{bmatrix} \cdot 2 + \begin{bmatrix} 3 \end{bmatrix} \cdot 3$$

Example :

$$AB = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix} = \begin{bmatrix} \ \end{bmatrix}$$

$$4\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 0\begin{bmatrix} 2 \\ 6 \end{bmatrix} + 2\begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$1\begin{bmatrix} 1 \\ 2 \end{bmatrix} - 1\begin{bmatrix} 2 \\ 6 \end{bmatrix} + 7\begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$4\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 3\begin{bmatrix} 2 \\ 6 \end{bmatrix} + 5\begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$3\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 1\begin{bmatrix} 2 \\ 6 \end{bmatrix} + 2\begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

Column-Row Expansion

$$\begin{array}{c} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & & \\ a_{m1} & a_{m2} & \cdots & a_{mk} \end{bmatrix} \\ \begin{array}{cccc} c_1 & c_2 & & c_k \end{array} \end{array} \quad \begin{array}{cc} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{k1} & b_{k2} & \cdots & b_{kn} \end{bmatrix} & \begin{array}{c} r_1 \\ r_2 \\ \\ r_k \end{array} \end{array}$$

$$AB = c_1 r_1 + c_2 r_2 + \cdots + c_k r_k$$

Example:

$$AB = \begin{array}{c} \begin{array}{cc} c_1 & c_2 \end{array} \\ \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} \end{array} \begin{array}{cc} \begin{bmatrix} 2 & 0 & 4 \\ -3 & 5 & 1 \end{bmatrix} & \begin{array}{c} r_1 \\ r_2 \end{array} \end{array}$$

$$= c_1 r_1 + c_2 r_2$$

$$= \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 2 & 0 & 4 \end{bmatrix} + \begin{bmatrix} 3 \\ -1 \end{bmatrix} \begin{bmatrix} -3 & 5 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 4 \\ 4 & 0 & 8 \end{bmatrix} + \begin{bmatrix} -9 & 15 & 3 \\ 3 & -5 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -7 & 15 & 7 \\ 7 & -5 & 7 \end{bmatrix}$$

Matrix Form of a Linear System

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad\qquad \vdots \qquad\qquad & \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

entry

entry

entry

$$\Rightarrow \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Can be viewed as a linear combination!

$$\Rightarrow \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$\Rightarrow \quad A\mathbf{x} = \mathbf{b}$$

coefficient matrix

Recall:

The augmented matrix:

$$[A \mid b] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

3/9 end.

Example Compute A^{20} where $A = \begin{bmatrix} 1 & -3 \\ 1 & 1 \end{bmatrix}$

(91 北科大電通)

$$A^2 = \begin{bmatrix} 1 & -3 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -3 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -2 & -6 \\ 2 & -2 \end{bmatrix}$$

$$A^3 = A^2 \cdot A = \begin{bmatrix} -2 & -6 \\ 2 & -2 \end{bmatrix} \cdot \begin{bmatrix} 1 & -3 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -8 & 0 \\ 0 & -8 \end{bmatrix} = (-8) \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = -8I$$

$$\therefore A^{20} = (A^3)^6 \cdot A^2 = (-8I)^6 \cdot A^2$$

$$= 8^6 (I^6 \cdot A^2)$$

$$= 8^6 (IA^2)$$

$$= 8^6 \cdot A^2 = \begin{bmatrix} -2(8^6) & -6(8^6) \\ 2(8^6) & -2(8^6) \end{bmatrix}$$

Example :

$$\text{Compute } X = \left[\begin{array}{cc|cc} 1 & 0 & 0 & -3 \\ 0 & 1 & 3 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]^{2007}$$

(96 大同資工)

Let $A = \begin{bmatrix} 0 & -3 \\ 3 & 0 \end{bmatrix}$, then $X = \begin{bmatrix} I_2 & A \\ 0 & I_2 \end{bmatrix}$

$I \cdot A + A \cdot I = A + A = 2A$

We find that $X^2 = \begin{bmatrix} I & A \\ 0 & I \end{bmatrix}\begin{bmatrix} I & A \\ 0 & I \end{bmatrix} = \begin{bmatrix} I & 2A \\ 0 & I \end{bmatrix}$

and $X^3 = XX^2 = \begin{bmatrix} I & A \\ 0 & I \end{bmatrix} \cdot \begin{bmatrix} I & 2A \\ 0 & I \end{bmatrix} = \begin{bmatrix} I & 3A \\ 0 & I \end{bmatrix}$

$\vdots$

$$\therefore X^{2007} = \begin{bmatrix} I & 2007A \\ 0 & I \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -6021 \\ 0 & 1 & 6021 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$M_{m \times n}$, $m=n$

Trace :

For a square matrix A, the trace of A is $tr(A) = \sum_{i=1}^{n} a_{ii}$

Example :

$$A = \begin{bmatrix} -1 & 2 & 7 & 0 \\ 3 & 5 & -8 & 4 \\ 1 & 2 & 7 & -3 \\ 4 & -2 & 1 & 0 \end{bmatrix}$$

$tr(A) = -1+5+7+0 = 11$

Upper/lower triangular matrix

A is a square matrix of order n and $a_{ij}=0$ $\forall i>j$ ($\forall i<j$) then A is an upper (lower) triangular matrix.

upper triangular matrix ; $A = \begin{bmatrix} a_{11} & & & * \\ & a_{22} & & \\ & & \ddots & \\ O & & & a_{nn} \end{bmatrix}$

lower triangular matrix

$$A = \begin{bmatrix} a_{11} & & & O \\ & a_{22} & & \\ & & \ddots & \\ * & & & a_{nn} \end{bmatrix}$$

If A is both upper and lower triangular, then A is called a diagonal matrix.

Example Suppose that A and B are symmetric matrices each of order n.
Prove that AB is a symmetric matrix if and only if

$AB = BA$. (91 師大資工)

$(\Rightarrow)$: AB is symmetric $\Rightarrow (AB)^T = AB$

$\Rightarrow B^T A^T = AB$

$\Rightarrow BA = AB$

$(\Leftarrow)$:

$\because AB = BA$

$\therefore (AB)^T = B^T A^T = BA = AB$

$\therefore AB$ is symmetric

Example (先訂立):

Prove that $(AB)^T = B^T A^T$ (95 南台資工 99 雲科大資工)

Suppose that $A \in \mathbb{F}^{m\times n}$, $B \in \mathbb{F}^{n\times p}$

$$((AB)^T)_{ij} = (AB)_{ji}$$

$$= \sum_{k=1}^{n} a_{jk} \cdot b_{ki}$$

$$= \sum_{k=1}^{n} b_{ki} \cdot a_{jk}$$

$$= \sum_{k=1}^{n} (B^T)_{ik} \cdot (A^T)_{kj}$$

$$= (B^T A^T)_{ij}$$

$\forall\, i = 1, 2, \ldots, p$, $j = 1, 2, \ldots, m$ $\therefore (AB)^T = B^T A^T$

Note: $AB \in \mathbb{F}^{m\times p}$

$(AB)^T \in \mathbb{F}^{p\times m}$

Theorem Suppose that $A, B \in \mathbb{F}^{n\times n}$, $\alpha, \beta \in \mathbb{F}$, then (3/15)

(1) $tr(\alpha A \pm \beta B) = \alpha \cdot tr(A) \pm \beta \cdot tr(B)$

(2) $tr(A^T) = tr(A)$

(3) $tr(I_n) = n$

$$tr(\overset{C}{\alpha A + \beta B}) = \sum_{i=1}^{n} c_{ii} = \sum_{i=1}^{n} ((\alpha A)_{ii} + (\beta B)_{ii}) = \sum_{i=1}^{n} (\alpha A)_{ii} + \sum_{i=1}^{n} (\beta B)_{ii} = tr(\alpha A) + tr(\beta B) = \alpha \cdot tr(A) + \beta \cdot tr(B)$$

Example Prove that there do not exist $n\times n$ matrices A and B such that $AB - BA = I_n$

(90中央數學, 91台大數學, 95彰師大資工)

(proof):

Suppose that there exist $A, B \in \mathbb{F}^{n\times n}$ such that

$$AB - BA = I_n$$

Then $tr(AB - BA) = tr(AB) - tr(BA) = tr(AB) - tr(AB) = 0$

But $tr(I_n) = n$

Theorem Suppose $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{n\times m}$, then $tr(AB) = tr(BA)$

(99台大電機, 95台科大資工, 95暨大通訊)

proof:

Suppose that $C = AB \in \mathbb{R}^{m\times m}$, $D = BA \in \mathbb{R}^{n\times n}$

then
$$tr(AB) = tr(C) = \sum_{i=1}^{m} c_{ii} = \sum_{i=1}^{m}\sum_{k=1}^{n} a_{ik} \cdot b_{ki} = \sum_{k=1}^{n}\sum_{i=1}^{m} b_{ki} \cdot a_{ik} = \sum_{k=1}^{n} d_{kk} = tr(D) = tr(BA)$$

Problem

1- A matrix B is said to be square root of a matrix A if $BB = A$

(1) Find two square roots of $A = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$

(2) How many different square roots can you find of

$$A = \begin{bmatrix} 5 & 0 \\ 0 & 9 \end{bmatrix}$$

(3) Do you think that every 2×2 matrix has at least one square root? Explain your reasoning.

Sol: (1)

Let $B = \begin{bmatrix} x & y \\ z & u \end{bmatrix}$, $B^2 = \begin{bmatrix} x^2+yz & xy+yu \\ xz+uz & yz+u^2 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$

$\therefore x^2+yz = u^2+yz = 2$

$\Rightarrow x^2 = u^2$

① if $x = -u$

$\Rightarrow xz+uz = 0 = 2$ ($\Rightarrow\Leftarrow$)

② we have $x = u$ $\Longrightarrow B^2 = \begin{bmatrix} u^2+z^2 & 2uz \\ 2uz & z^2+u^2 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$

$\Rightarrow xy+yu = 2yu = 2 \Rightarrow u = z$ $\quad \therefore u^2+z^2 = 2uz = 2$

$xz+uz = 2uz = 2$ $\quad \Rightarrow (u-z)^2 = 0$

and $yu = uz = 1 \Rightarrow y = z$ ($\because u \neq 0$) $\quad u = z$ or $u = -z$

$xy =$

$B^2 = \begin{bmatrix} x & y \\ x & x \end{bmatrix}\begin{bmatrix} x & y \\ x & x \end{bmatrix} = \begin{bmatrix} x^2+xy & 2xy \\ 2x^2 & x^2+xy \end{bmatrix}$ $\quad B^2 = \begin{bmatrix} 2z^2 & 2z^2 \\ 2z^2 & 2z^2 \end{bmatrix}$

$\therefore x = \pm 1$, $y = \pm 1$

or $B^2 = \begin{bmatrix} 2z^2 & -2z^2 \\ -2z^2 & 2z^2 \end{bmatrix}$

$B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, or $\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}$ Verify them.

$\therefore uz = 1$

$\therefore u = z = 1$

or $u = z = -1$

(2) Let $B = \begin{bmatrix} x & y \\ z & u \end{bmatrix}$

$$B^2 = \begin{bmatrix} x^2+yz & xy+yu \\ xz+uz & yz+u^2 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 9 \end{bmatrix}$$

$$\Rightarrow \begin{cases} y(x+u) = 0 \\ z(x+u) = 0 \end{cases}$$

① if $y = 0 \Rightarrow B^2 = \begin{bmatrix} x & 0 \\ z & u \end{bmatrix}\begin{bmatrix} x & 0 \\ z & u \end{bmatrix} = \begin{bmatrix} x^2 & 0 \\ xz+uz & u^2 \end{bmatrix}$

$x^2 = 5 \Rightarrow x = \pm\sqrt{5}$

$u^2 = 9 \Rightarrow u = \pm 3$

$\Rightarrow$ That means $x+u \neq 0 \quad \therefore z = 0$

We have $B = \begin{bmatrix} \pm\sqrt{5} & 0 \\ 0 & \pm 3 \end{bmatrix}$ (4 square roots of A)

(3) Try to find a square root of $A = \begin{bmatrix} 0 & 9 \\ 0 & 0 \end{bmatrix}$

Let $B = \begin{bmatrix} x & y \\ z & u \end{bmatrix}$

$$B^2 = \begin{bmatrix} x^2+yz & xy+yu \\ xz+uz & u^2+yz \end{bmatrix} = \begin{bmatrix} 0 & 9 \\ 0 & 0 \end{bmatrix}$$

$\Rightarrow x^2+yz = u^2+yz = 0$

$\Rightarrow x^2 = u^2 = 0 \Rightarrow x = u = 0$

$\therefore xy+yu = 0+0 = 0 \neq 9$ (⇒⇐)

Problem For any two 2×2 matrices A and B, is "$AB=BA$" always true? why?

(sol): At the beginning, $\forall i,j \in \{1,2\}$, $(AB)_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj} = ?$

Find an example (counter-example):

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ 20 & 13 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}\cdot\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = ?$$

Problem: If A, B, and C are all $n\times n$ square matrices, such that $AC=BC$, then $A=B$.

Is that always true? Why?

(sol): It seems that $AC=BC \Rightarrow A\cancel{C} = B\cancel{C} \Rightarrow A=B$?

Let's find a counter-example?

Let C be $\begin{bmatrix} 0 & 9 \\ 0 & 0 \end{bmatrix}$,

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 7 \\ 3 & 8 \end{bmatrix}$

Then $AC = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 0 & 9 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 9 \\ 0 & 27 \end{bmatrix}$ WTF?!

$BC = \begin{bmatrix} 1 & 7 \\ 3 & 8 \end{bmatrix}\begin{bmatrix} 0 & 9 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 9 \\ 0 & 27 \end{bmatrix}$