Inverses

Properties of Matrix Arithmetic

Assume that the sizes of the matrices allow the following operations to be performed.

The following rules of matrix arithmetic are valid:

(a) $A+B = B+A$ (commutative law)

(b) $A+(B+C) = (A+B)+C$ (associative law for matrix addition)

(c) $A(BC) = (AB)C$ (associative law for matrix multiplication)

使得括號不需要

(d) $A(B+C) = AB+AC$ (distributive law)

(e) $(B+C)A = BA+CA$ (")

(f) $A(B-C) = AB-AC$

(g) $(B-C)A = BA-CA$

(h) $a(B\pm C) = aB \pm aC$, $\forall a \in \mathbb{F}$

(i) $(a\pm b)A = aA \pm bA$, $\forall a, b \in \mathbb{F}$

(j) $a(bA) = (ab)A$

(k) $a(BC) = (aB)C = B(aC)$

a proof for (d): Let $A \in \mathbb{F}^{m\times n}$, $B, C \in \mathbb{F}^{n\times r}$

$$
\begin{aligned}
(A(B+C))_{ij} &= \sum_{k=1}^{n} a_{ik}(B+C)_{kj} \\
&= \sum_{k=1}^{n} a_{ik}(b_{kj}+c_{kj}) \\
&= \sum_{k=1}^{n} (a_{ik}b_{kj} + a_{ik}c_{kj}) \\
&= \sum_{k=1}^{n} a_{ik}b_{kj} + \sum_{k=1}^{n} a_{ik}c_{kj} \\
&= (AB)_{ij} + (AC)_{ij} \\
&= (AB+AC)_{ij}
\end{aligned}
$$

Note:

In $\mathbb{F}$, $\forall a, b \in \mathbb{F}$, $a \cdot b = b \cdot a$

But in matrix arithmetic,

$AB = BA$ is NOT always true.

For example, ① $A \in \mathbb{F}^{2\times 3}$ but $B \in \mathbb{F}^{3\times 4}$

BA is NOT defined!

② $A \in \mathbb{F}^{2\times 3}$, $B \in \mathbb{F}^{3\times 2}$

Then $AB \in \mathbb{F}^{2\times 2}$

but $BA \in \mathbb{F}^{3\times 3}$

③ Even the sizes are the same, AB and BA are defined,

$AB = BA$ is NOT always true.

example:

$$A = \begin{bmatrix} -1 & 0 \\ 2 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}$$

$$\Rightarrow AB = \begin{bmatrix} -1 & -2 \\ 11 & 4 \end{bmatrix}, \quad BA = \begin{bmatrix} 3 & 6 \\ -3 & 0 \end{bmatrix}$$

$$AB \neq BA$$

Zero matrix:

$$0^{m\times n}$$

If $A \in \mathbb{F}^{m\times n}$, then $A + 0 = 0 + A = A$

$0^{m\times n}$: 加法單位元素

Properties of Zero Matrices

(a) $A + 0 = 0 + A = A$

(b) $A - 0 = A$

(c) $A - A = A + (-A) = 0$

(d) $0A = 0$

(e) if $cA = 0$, then $c = 0$ or $A = 0$

Note

In $\mathbb{F}$, $\forall a, b, c \in \mathbb{F}$,

① If $ab = ac$ and $a \neq 0$ then $b = c$ (Cancellation law)

② If $ab = 0$, then either a or $b = 0$

But they are NOT always true in matrix arithmetic!

Example

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 5 \\ 3 & 4 \end{bmatrix}$$

We have $AB = \begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix}$

$AC = \begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix}$

$A \neq 0$

So $B = C$??

Example For $A = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 7 \\ 0 & 0 \end{bmatrix}$,

$AB = 0$ but $A \neq 0$ and $B \neq 0$!!!

Identity Matrices I_n

$$I_n = \begin{bmatrix} a_{11} & & 0 \\ & a_{22} & \\ 0 & \ddots & a_{nn} \end{bmatrix}, \quad a_{11} = a_{22} = \cdots = a_{nn} = 1$$

Try to calculate : AI_3, where $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$

$$AI_3 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}_{2\times 3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3}$$

$$= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = A$$

Also,

$$I_2 A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times 2} \cdot \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}_{2\times 3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = A$$

$\therefore$ For $A \in \mathbb{F}^{m\times n}$

$$AI_n = A, \quad I_m A = A$$

Inverse of a Matrix

In $\mathbb{F}$, $\forall a \in \mathbb{F}$, $a^{-1} \in \mathbb{F}$ such that $a \cdot (a^{-1}) = (a^{-1}) \cdot a = 1$

$\Rightarrow$ it's the multiplicative inverse of a

Definition :

If A is a **square** matrix, and if there exists a matrix B of the same size for which $AB = BA = I$, then A is said to be **invertible** (or nonsingular) and B is called an **inverse** of A.

If no such matrix B exists, then we say A is **singular**

Example:

$$A = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$$

$$AB = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

$$BA = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

$\therefore$ A and B are both invertible and each is an inverse of the other.

Example

Any square matrix with a row or columns of zeros is singular.

Let $A = \begin{bmatrix} 1 & 4 & 0 \\ 2 & 5 & 0 \\ 3 & 6 & 0 \end{bmatrix}$. We want to show that there is no $B \in \mathbb{F}^{3\times 3}$

such that $AB = BA = I$

$A = [C_1, C_2, 0]$ where $C_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $C_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$

$\therefore \forall B \in \mathbb{F}^{3\times 3}$, $BA = B[C_1, C_2, \mathbf{0}] = [BC_1, BC_2, 0] \neq I$

?!

$\therefore$ A is singular

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \\ 0 \end{bmatrix}$, where $r_1 = [1, 2, 3]$, $r_2 = [4\ 5\ 6]$

$\therefore \forall B \in \mathbb{F}^{3\times 3}$, $AB = \begin{bmatrix} r_1 \\ r_2 \\ 0 \end{bmatrix} B = \begin{bmatrix} r_1 B \\ r_2 B \\ \mathbf{0} \end{bmatrix} \neq I$

?!

Properties of Inverses

Theorem If B and C are both inverses of matrix A,

then $B = C$

An Inverse Is Unique!

(proof):

$\because$ B is an inverse of A,

$\therefore$ $BA = I$

$\Rightarrow (BA)C = IC = C$ ……①

But $(BA)C = B(AC)$ and since C is an inverse of A,

$\Rightarrow AC = I$

$\therefore (BA)C = B(AC) = B \cdot I = B$ ‥②

Thus, $B = C$ $\#$

$\therefore$ A^{-1} : "the" inverse of A.

Note : A^{-1} should not be interpreted as $\frac{1}{A}$

Theorem For a matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{F}^{2\times 2}$

determinant 行列式值

A is invertible if and only if $ad - bc \neq 0$

and $A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

↪ 1858年, Arthur Cayley 首次提出.

Example: (a) $A = \begin{bmatrix} 6 & 1 \\ 5 & 2 \end{bmatrix}$, (b) $A = \begin{bmatrix} -1 & 2 \\ 3 & -6 \end{bmatrix}$

(a):

$$\det(A) = 6 \cdot 2 - 1 \cdot 5 = 7 \neq 0$$

$$\therefore A^{-1} = \frac{1}{7}\begin{bmatrix} 2 & -1 \\ -5 & 6 \end{bmatrix} = \begin{bmatrix} \frac{2}{7} & -\frac{1}{7} \\ -\frac{5}{7} & \frac{6}{7} \end{bmatrix}$$

(b)

$$\det(A) = (-1)\cdot(-6) - 2 \cdot 3 = 0$$

$\therefore$ A is NOT invertible.

Example: Solve $\begin{cases} u = ax + by \\ v = cx + dy \end{cases}$ for x and y in terms of u and v.

Method 1: Gauss-Jordan elimination!

ORZ

Method 2:

Rewrite it as $\begin{bmatrix} u \\ v \end{bmatrix} = \underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \end{bmatrix}$

$$\therefore A^{-1}\begin{bmatrix} u \\ v \end{bmatrix} = A^{-1}A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \frac{du-bv}{ad-bc} \\ \frac{-cu+av}{ad-bc} \end{bmatrix}$$

Theorem If $A, B \in \mathbb{F}^{n\times n}$ are invertible, then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$

proof):

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$$

Similarly,

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I$$

$\therefore$ AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$

Powers of a Matrix

① If A is a square matrix, then we define the nonnegative powers of A to be

$$A^0 = I$$

$$A^n = \underbrace{A \cdot A \cdots A}_{n \text{ times}}$$

② If A is invertible, then

$$A^{-n} = (A^{-1})^n = \underbrace{(A^{-1})\cdot(A^{-1})\cdots A^{(-1)}}_{n \text{ times}}$$

③ $A^rA^s = A^{r+s}$ and $(A^r)^s = A^{rs}$ for integers r, s

Theorem If A is invertible and n is a nonnegative integer, then:

(a) A^{-1} is invertible and $(A^{-1})^{-1} = A$

(b) A^{n} is invertible and $(A^{n})^{-1} = A^{-n} = (A^{-1})^{n}$

(c) kA is invertible for any $k \in \mathbb{F}$, $k > 0$

(proof):

(c)

$$(kA)(k^{-1}A^{-1}) = k^{-1}(kA)A^{-1} = (k^{-1}k)AA^{-1} = (1\cdot)I = I$$

(a) $(A^{-1})\cdot A = I$ $\therefore$ A is in the inverse of A^{-1}

$A(A^{-1}) = I$ $\Rightarrow A = (A^{-1})^{-1}$

Example: Compute A^{3} and A^{-3} where $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$

$$A^{-1} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$$

$$A^{-3} = (A^{-1})^{3} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 41 & -30 \\ -15 & 11 \end{bmatrix}$$

$$A^{3} = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 8 \\ 4 & 11 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 30 \\ 15 & 41 \end{bmatrix}$$

$$A^{-3} = (A^{-1})^{3} = \begin{bmatrix} 41 & -30 \\ -15 & 11 \end{bmatrix}$$

$$(A^{3})^{-1} = \frac{1}{11\cdot 41 - 30\cdot 15}\begin{bmatrix} 41 & -30 \\ -15 & 11 \end{bmatrix} = \begin{bmatrix} 41 & -30 \\ -15 & 11 \end{bmatrix}$$

$$= A^{-3}$$

Square of a Matrix Sum

$$(A+B)^2 = (A+B)(A+B) = A^2 + \underbrace{BA + AB}_{\overset{(\times)}{\Updownarrow}\ 2AB} + B^2$$

Only when $AB = BA$

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Matrix polynomials

If $A \in \mathbb{F}^{n\times n}$, $p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_m x^m$

Then we define the $n\times n$ matrix $P(A)$ as

$$p(A) = a_0 I + a_1 A + a_2 A^2 + \cdots + a_m A^m$$

the matrix polynomial in A.

Example: $p(x) = x^2 - 2x - 5$ and $A = \begin{bmatrix} -1 & 2 \\ 1 & 3 \end{bmatrix}$ 把矩陣整個當一個變數

$$\therefore\ p(A) = A^2 - 2A - 5I$$

$$= \begin{bmatrix} -1 & 2 \\ 1 & 3 \end{bmatrix}^2 - 2\cdot\begin{bmatrix} -1 & 2 \\ 1 & 3 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 4 \\ 2 & 11 \end{bmatrix} - \begin{bmatrix} -2 & 4 \\ 2 & 6 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\therefore\ P(A) = 0$$

Recall: properties of the transpose

(a) $(A^T)^T = A$.

(b) $(A \pm B)^T = A^T \pm B^T$

(c) $(kA)^T = k \cdot A^T$

(d) $(AB)^T = B^T A^T$

Theorem: If A is invertible, then A^T is also invertible and $(A^T)^{-1} = (A^{-1})^T$

proof: $(A^T)(A^{-1})^T = (A^{-1}A)^T = I^T = I$

and $(A^{-1})^T \cdot A^T = (AA^{-1})^T = I^T = I$ ☆

Example:

Consider $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$

Assume that A is invertible, $ad-bc \neq 0$

$\det(A^T) = ad - bc \neq 0$, so A^T is also invertible

$$(A^T)^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$$

Also, $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$$\Rightarrow (A^{-1})^T = \frac{1}{ad-bc}\begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$$

Matrix Polynomials (重覆～ 跳過)

$A \in \mathbb{F}^{n\times n}$

if $p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_m x^m$ is a polynomial

We define the <u>$n\times n$ matrix $p(A)$</u>:

$$p(A) = a_0 I_n + a_1 A + a_2 A^2 + \cdots + a_m A^m \quad \cdots\cdots (*)$$

$(*)$ is called a matrix polynomial in A.

Example: Find $p(A)$ for $p(x) = x^2 - 2x - 5$ and $A = \begin{bmatrix} -1 & 2 \\ 1 & 3 \end{bmatrix}$

(sol):

$$\begin{aligned} p(A) &= A^2 - 2A - 5I \\ &= \begin{bmatrix} -1 & 2 \\ 1 & 3 \end{bmatrix}^2 - 2\begin{bmatrix} -1 & 2 \\ 1 & 3 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 4 \\ 2 & 11 \end{bmatrix} + \begin{bmatrix} 2 & -4 \\ -2 & -6 \end{bmatrix} + \begin{bmatrix} -5 & 0 \\ 0 & -5 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ &= \mathbf{0} \end{aligned}$$

Note: For any polynomials p_1 and p_2, we have

$$p_1(A) \cdot p_2(A) = p_2(A) \cdot p_1(A)$$

You can check that by yourselves, using the fact that

$$A^r A^s = A^{r+s} = A^{s+r} = A^s A^r$$

Problem : The Fibonacci sequence (Leonardo Fibonacci 1170 ~ 1250)

$$\begin{array}{cccccccccccccc} 0, & 1, & 1, & 2, & 3, & 5, & 8, & 13, & 21, & 34, & 55, & 89, & 144, & \dots \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & & & & & \\ F_0 & F_1 & F_2 & F_3 & F_4 & F_5 & F_6 & F_7 & & & & & & \end{array}$$

After the initial terms $F_0 = 0$ and $F_1 = 1$, each term is the sum of the previous two :

$$F_n = F_{n-1} + F_{n-2}$$

Confirm that if $Q = \begin{bmatrix} F_2 & F_1 \\ F_1 & F_0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

then $Q^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}$

(sol) Using mathematical induction :

$$Q^1 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} F_2 & F_1 \\ F_1 & F_0 \end{bmatrix} \checkmark$$

Suppose that (inductive hypothesis)

$$Q^k = \begin{bmatrix} F_{k+1} & F_k \\ F_k & F_{k-1} \end{bmatrix}$$

Then $Q^{k+1} = Q^k \cdot Q$ (inductive step)

$$= \begin{bmatrix} F_{k+1} & F_k \\ F_k & F_{k-1} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} F_{k+1} + F_k & F_{k+1} \\ F_k + F_{k-1} & F_k \end{bmatrix}$$

$$= \begin{bmatrix} F_{k+2} & F_{k+1} \\ F_{k+1} & F_k \end{bmatrix}$$

$\therefore Q^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}$ is true for all $n \geq 1$ ☒