Elementary Matrices

We have learned elementary row operations ...

$\Rightarrow$ Say $e_1, e_2, \cdots$

$$A \xrightarrow{e_1} A_1 \xrightarrow{e_2} A_2 \xrightarrow{e_3} \cdots \xrightarrow{e_k} B \quad ?!$$

$$A \xleftarrow{e_1^{-1}} A_1 \xleftarrow{e_2^{-1}} A_2 \xleftarrow{e_3^{-1}} \cdots \xleftarrow{e_k^{-1}} B$$

We say matrices A and B are **row equivalent** if each of them can be obtained from the other by a sequence of elementary row operations.

Definition

A matrix E is called an elementary matrix if it can be obtained from an identity matrix by performing a single elementary row operation.

e.g.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \to (\times 3) \Rightarrow E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times 3 \Rightarrow E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$$

Example: Using elementary matrices = elementary row operations

Consider $A = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 1 & 4 & 4 & 0 \end{bmatrix}$ and consider $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$

$$EA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 1 & 4 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 4 & 4 & 10 & 9 \end{bmatrix}$$

equivalent to $\begin{bmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 1 & 4 & 4 & 0 \end{bmatrix} \times 3$

$$I = \begin{array}{c} \text{row } i \\ \\ \end{array}\begin{bmatrix} & & \\ & & \end{bmatrix} \times c \longrightarrow E : \begin{array}{c} \text{row } i \\ \\ \end{array}\begin{bmatrix} & & \\ & & \end{bmatrix} \times \frac{1}{c}$$

$$R_i^{(c)}$$

$$I = \begin{array}{c} \text{row } i \\ \text{row } j \end{array}\begin{bmatrix} & & \\ & & \end{bmatrix} \text{interchange} \longrightarrow E : \begin{array}{c} \text{row } i \\ \text{row } j \end{array}\begin{bmatrix} & & \\ & & \end{bmatrix} \text{interchange}$$

$$R_{ij}$$

$$I = \begin{array}{c} \text{row } i \\ \text{row } j \end{array}\begin{bmatrix} & & \\ & & \end{bmatrix} \times c \longrightarrow E : \begin{array}{c} \text{row } i \\ \text{row } j \end{array}\begin{bmatrix} & & \\ & & \end{bmatrix} \times (-c)$$

$$R_{ij}^{(c)}$$

Example

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{\times 7} \longrightarrow \begin{bmatrix} 1 & 0 \\ 0 & 7 \end{bmatrix}_{\times \frac{1}{7}} \longrightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Theorem: Every elementary matrix is invertible.
The inverse is also an elementary matrix.

$$E^{-1} \cdot E = I, \quad E \cdot E^{-1} = I$$

<u>Theorem</u> If $A \in F^{n\times n}$, then the following statements are equivalent:

(a) A is invertible

(b) $A\mathbf{x} = 0$ has only the trivial solution.

(c) The reduced row echelon form of A is I_n

(d) A is expressible as a product of elementary matrices.

(a)$\Rightarrow$(b)

Assume that A is invertible and let $\mathbf{x}_0$ be any solution of $A\mathbf{x} = 0$

$$\Rightarrow (A^{-1}A)\mathbf{x}_0 = A^{-1}0$$

$$\Rightarrow I\cdot\mathbf{x}_0 = 0$$

$$\Rightarrow \mathbf{x}_0 = 0 \Rightarrow A\mathbf{x} = 0 \text{ has only the trivial solution.}$$

(b)$\Rightarrow$(c):

The matrix form of the system $A\mathbf{x} = \mathbf{0}$:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= 0 \end{aligned}$$

Assume that it has only the trivial solution

By Gauss-Jordan elimination, the reduced row echelon form:

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & \cdots & & 1 & 0 \end{array}\right] \Leftrightarrow \begin{aligned} x_1 \quad\quad &= 0 \\ x_2 \quad &= 0 \\ \ddots \quad & \\ x_n &= 0 \end{aligned}$$

$\therefore$ The reduced row echelon form is I_n

(c) $\Rightarrow$ (d): Assume that the reduced row echelon form is I_n

$\therefore$ We can find elementary matrices $E_1, E_2, \cdots, E_k$ such that

$E_k E_{k-1} \cdots E_2 E_1 A = I_n$ the same

Since elementary matrices are invertible,

$\therefore (E_1^{-1} \cdot E_2^{-1} \cdots \cdot E_k^{-1})(E_k E_{k-1} \cdots E_2 \cdot E_1 A) = E_1^{-1} \cdot E_2^{-1} \cdots \cdot E_k^{-1}$

$\Rightarrow A = E_1^{-1} E_2^{-1} \cdots E_k^{-1} I \qquad \therefore \underbrace{E_k \cdots E_3 E_2 E_1}_{A^{-1}} A = I$

$A = I E_1^{-1} E_2^{-1} \cdots E_k^{-1} \qquad \therefore A \underbrace{E_k E_{k-1} \cdots E_2 E_1}_{A^{-1}} = I$

(d) $\Rightarrow$ (a):

If A is a product of elementary matrices,

then A is a product of invertible matrices

$\Rightarrow A$ is invertible ☆

Therefore, we are aware of a method of finding an inverse of a matrix!

$$A^{-1} = E_k E_{k-1} \cdots E_2 \cdot E_1 \cdot I_n$$

$\Rightarrow$ Here comes the Inversion Algorithm:

Example: $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$

We adjoin the identity matrix I_3 to the right side of A: $[A|I]$

The goal:

$$[A \mid I] \rightsquigarrow [I \mid A^{-1}]$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 9 & 5 & -2 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 9 & 5 & -2 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

$$\therefore A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix} \quad ☆$$

<u>Example</u> $A = \begin{bmatrix} 1 & 6 & 4 \\ 2 & 4 & -1 \\ -1 & 2 & 5 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 2 & 4 & -1 & 0 & 1 & 0 \\ -1 & 2 & 5 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \times(-2) \\ \\ \times 1 \end{array}$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 0 & -8 & -9 & -2 & 1 & 0 \\ 0 & 8 & 9 & 1 & 0 & 1 \end{array}\right] \times(1)$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 0 & -8 & -9 & -2 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 \end{array}\right]$$

$\Rightarrow$ the reduced row echelon form is NOT I_3 !!

A is NOT invertible $\star$

<u>Example</u>: Find the inverse of the product

$$A = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -c & 1 \end{bmatrix}}_{W_1} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -b & 0 & 1 \end{bmatrix}}_{W_2} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{W_3}$$

(91 中興 CSIE)

(Sol):

We observe that W_1, W_2, W_3 are elementary matrices

$\therefore$ W_1, W_2, W_3 are invertible

i.e., $A = R_{23}^{(-c)} R_{13}^{(-b)} R_{12}^{(-a)}$

$$\therefore A^{-1} = \left(R_{23}^{(-c)} R_{13}^{(-b)} R_{12}^{(-a)}\right)^{-1}$$

$$= \left(R_{12}^{(-a)}\right)^{-1} \left(R_{13}^{(-b)}\right)^{-1} \left(R_{23}^{(-c)}\right)^{-1}$$

$$= R_{12}^{(a)} R_{13}^{(b)} R_{23}^{(c)}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ b & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix} \#$$

Example Let $A = \begin{bmatrix} 1 & 0 \\ -5 & 2 \end{bmatrix}$. Find elementary matrices E_1 and E_2 such that $A = E_1 E_2$ (98 RU (SIE))

(sol):

$$A = \begin{bmatrix} 1 & 0 \\ -5 & 2 \end{bmatrix} \xrightarrow{r_{12}^{(5)}} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \xrightarrow{r_2^{(\frac{1}{2})}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$\therefore R_2^{(\frac{1}{2})} R_{12}^{(5)} A = I$$

$$\Rightarrow A = \left(R_2^{(\frac{1}{2})} R_{12}^{(5)}\right)^{-1}$$

$$= R_{12}^{(-5)} \cdot R_2^{(2)}$$

$$= \begin{bmatrix} 1 & 0 \\ -5 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\therefore E_1 = \begin{bmatrix} 1 & 0 \\ -5 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \quad \#$$

$$B = R_2^{(\frac{1}{2})} R_{12}^{(5)} R_{34}(A)$$

$$\therefore \left(R_2^{(\frac{1}{2})} R_{12}^{(5)} R_{34}\right)^{-1} B = A$$

$$R_{34} R_{12}^{(-5)} R_2^{(2)}$$

Theorem A system of linear equations has zero, one, or infinitely many solutions. There are NO other possibilities.

proof):

If $A\mathbb{X} = b$ is a system of linear equations. Then exactly one of the following is true:

(a) the system has no solutions

(b) the system has exactly one solution

✓ (c) the system has >1 solution

Let $\mathbb{X}_1, \mathbb{X}_2$ be any two distinct solutions of $A\mathbb{X} = b$

$\mathbb{X}_0 = \mathbb{X}_1 - \mathbb{X}_2 \neq 0$

$\therefore A\mathbb{X}_0 = A(\mathbb{X}_1 - \mathbb{X}_2) = A\mathbb{X}_1 - A\mathbb{X}_2 = b - b = 0$

if we let $k \in \mathbb{F}$ be any scalar, then

$$A(\mathbb{X}_1 + k\mathbb{X}_0) = A\mathbb{X}_1 + A(k\mathbb{X}_0) = A\mathbb{X}_1 + k(A\mathbb{X}_0)$$
$$= b + k\cdot 0 = b + 0 = b$$

So $\mathbb{X}_1 + k\mathbb{X}_0$ is a solution of $A\mathbb{X} = b$.

Since $\mathbb{X}_0 \neq 0$, there are infinitely many choices for k.

$\therefore A\mathbb{X} = b$ has infinitely many solutions.

Theorem If $A \in \mathbb{F}^{n\times n}$, A is invertible, then

for each $b \in \mathbb{F}^{n\times 1}$, $A\mathbf{x} = b$ has exactly one solution

$\mathbf{x} = A^{-1}b$

(proof): $\because A(A^{-1}b) = b$

$\therefore \mathbf{x} = A^{-1}b$ is a solution of $A\mathbf{x} = b$

Assume that $\mathbf{x}_0$ is an arbitrary solution of $A\mathbf{x} = b$

$\Rightarrow A\mathbf{x}_0 = b$

$\Rightarrow A^{-1}(A\mathbf{x}_0) = A^{-1}b$

$\Rightarrow \mathbf{x}_0 = A^{-1}b = \mathbf{x}$

Example

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 5 \\ 2x_1 + 5x_2 + 3x_3 &= 3 \\ x_1 \qquad\quad + 8x_3 &= 17 \end{aligned}$$

$$\Rightarrow A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}, \quad b = \begin{bmatrix} 5 \\ 3 \\ 17 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Find $A^{-1} \Rightarrow A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$

$$\therefore \mathbf{x} = A^{-1}b = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ 17 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

$\therefore x_1 = 1,\ x_2 = -1,\ x_3 = 2$

Linear Systems with a Common Coefficient Matrix

Example :

$A\mathbf{x} = b_1,\ A\mathbf{x} = b_2,\ A\mathbf{x} = b_3, \cdots, A\mathbf{x} = b_k$

$\Rightarrow \mathbf{x}_1 = A^{-1}b_1,\ \mathbf{x}_2 = A^{-1}b_2, \cdots, \mathbf{x}_k = A^{-1}b_k$

An efficient way of expression:

$[A \mid b_1 \mid b_2 \mid \cdots \mid b_k]$

Example

(a)
$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 4 \\ 2x_1 + 5x_2 + 3x_3 &= 5 \\ x_1 \qquad + 8x_3 &= 9 \end{aligned}$$

(b)
$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 1 \\ 2x_1 + 5x_2 + 3x_3 &= 6 \\ x_1 \qquad + 8x_3 &= -6 \end{aligned}$$

(Sol):

$$\left[\begin{array}{ccc|cc} 1 & 2 & 3 & 4 & 1 \\ 2 & 5 & 3 & 5 & 6 \\ 1 & 0 & 8 & 9 & -6 \end{array}\right]
\xRightarrow{r_{12}^{(-2)}}
\left[\begin{array}{ccc|cc} 1 & 2 & 3 & 4 & 1 \\ 0 & 1 & -3 & -3 & 4 \\ 1 & 0 & 8 & 9 & -6 \end{array}\right]$$

$$\xRightarrow{r_{13}^{(-1)}}
\left[\begin{array}{ccc|cc} 1 & 2 & 3 & 4 & 1 \\ 0 & 1 & -3 & -3 & 4 \\ 0 & -2 & 5 & 5 & -7 \end{array}\right]$$

$$\xRightarrow{r_{21}^{(-2)},\ r_{23}^{(2)}}
\left[\begin{array}{ccc|cc} 1 & 0 & 9 & 10 & -7 \\ 0 & 1 & -3 & -3 & 4 \\ 0 & 0 & -1 & -1 & 1 \end{array}\right]
\xRightarrow{r_3^{(-1)}}
\left[\begin{array}{ccc|cc} 1 & 0 & 9 & 10 & -7 \\ 0 & 1 & -3 & -3 & 4 \\ 0 & 0 & 1 & 1 & -1 \end{array}\right]$$

$$\xRightarrow{r_{32}^{(3)},\ r_{31}^{(9)}}
\left[\begin{array}{ccc|cc} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & -1 \end{array}\right]$$

$\therefore$ (a): $x_1 = 1,\ x_2 = 0,\ x_3 = 1$

(b): $x_1 = 2,\ x_2 = 1,\ x_3 = -1$

Equivalence Theorem

If $A \in \mathbb{F}^{n\times n}$ and ~~A is invertible~~, then the following are equivalent

(a) A is invertible

(b) $A\mathbf{x} = 0$ has only the trivial solution

(c) The reduced row echelon form of $A = I_n$

(d) A is expressible as a product of elementary matrices

NEW!

(e) $A\mathbf{x} = b$ is consistent for every $n\times 1$ matrix b

(f) $A\mathbf{x} = b$ has exactly one solution for every $n \times 1$ matrix b.

Try (e) $\Rightarrow$ (a): $A\mathbf{x} = b$ is consistent for every $n\times 1$ matrix b.

So, $A\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, $A\mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, $\dots$, $A\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$

Let $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$ be solutions of the above systems.

Let $C \in \mathbb{F}^{n\times n}$ be $C = [\mathbf{x}_1 \mid \mathbf{x}_2 \mid \mathbf{x}_3 \mid \cdots \mid \mathbf{x}_n]$

$\therefore \underset{\text{square}}{A}\,\underset{\text{square}}{C} = [A\mathbf{x}_1 \mid A\mathbf{x}_2 \mid \cdots \mid A\mathbf{x}_n] = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & 0 & & \vdots \\ \vdots & \vdots & & \\ 0 & 0 & \cdots & 1 \end{bmatrix} = I$

$\Rightarrow C = A^{-1}$, thus A is invertible #

Note:

Theorem 1.6.3 Let A be a square matrix.

(a) If B is a square matrix satisfying $BA = I$, then $B = A^{-1}$

(b) If B is a square matrix satisfying $AB = I$, then $B = A^{-1}$

(proof) (a): Goal: show that A is invertible $\Rightarrow$ it suffices to show that $A\mathbf{x} = 0$ has only the trivial solution

Let $\mathbf{x}_0$ be any solution of the system $\therefore A\mathbf{x}_0 = 0 \Rightarrow B(A\mathbf{x}_0) = B0 = 0$
$\Rightarrow I\mathbf{x}_0 = 0 \Rightarrow \mathbf{x}_0 = 0$

OK! Now we know A is invertible

Hence, $BA = I \Rightarrow BAA^{-1} = IA^{-1} \Rightarrow BI = IA^{-1} \Rightarrow B = A^{-1}$

(b) Goal: show that B is invertible, so that $AB = I \Rightarrow ABB^{-1} = B^{-1} \Rightarrow A = B^{-1}$

$\therefore A^{-1} = (B^{-1})^{-1} = B$

We can determine consistency of the system of equations by elimination! Especially when $A \in \mathbb{F}^{m \times n}$ is NOT invertible!

Example :

What conditions must b_1, b_2, and b_3 satisfy in order for the system of equations :

$$x_1 + x_2 + 2x_3 = b_1$$
$$x_1 + x_3 = b_2$$
$$2x_1 + x_2 + 3x_3 = b_3$$

to be consistent ?

sol):

The augmented matrix is

$$\begin{bmatrix} 1 & 1 & 2 & b_1 \\ 1 & 0 & 1 & b_2 \\ 2 & 1 & 3 & b_3 \end{bmatrix} \quad \times(-1)\ r_{12}^{(-1)} \quad \times(-2)\ r_{13}^{(-2)}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 2 & b_1 \\ 0 & -1 & -1 & b_2 - b_1 \\ 0 & -1 & -1 & b_3 - 2b_1 \end{bmatrix} \to \times(-1)\ r_2^{(-1)}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 2 & b_1 \\ 0 & 1 & 1 & b_1 - b_2 \\ 0 & -1 & -1 & b_3 - 2b_1 \end{bmatrix} \quad \times 1\ r_{23}^{(1)}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 2 & b_1 \\ 0 & 1 & 1 & b_1 - b_2 \\ 0 & 0 & 0 & b_3 - b_2 - b_1 \end{bmatrix}$$

$\therefore$ The system has a solution iff $b_3 - b_2 - b_1 = 0$

$$\Updownarrow$$

$$b_3 = b_1 + b_2$$

$$\therefore \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_1 + b_2 \end{bmatrix}, \quad \text{for } b_1, b_2 \in \mathbb{F}$$

Example

What conditions must b_1, b_2, and b_3 satisfy in order for the system of equations:

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= b_1 \\ 2x_1 + 5x_2 + 3x_3 &= b_2 \\ x_1 \qquad\quad + 8x_3 &= b_3 \end{aligned}$$

to be consistent?

(sol): The augmented matrix is :

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 2 & 5 & 3 & b_2 \\ 1 & 0 & 8 & b_3 \end{array}\right] \quad r_{12}^{(-2)} \quad r_{13}^{(-1)}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 0 & 1 & -3 & b_2-2b_1 \\ 0 & -2 & 5 & b_3-b_1 \end{array}\right] \quad r_{21}^{(-2)} \quad r_{23}^{(2)}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 9 & 5b_1-2b_2 \\ 0 & 1 & -3 & b_2-2b_1 \\ 0 & 0 & -1 & b_3+2b_2-5b_1 \end{array}\right] \qquad 2b_2 - 4b_1 + b_3 - b_1$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 9 & 5b_1-2b_2 \\ 0 & 1 & -3 & b_2-2b_1 \\ 0 & 0 & 1 & 5b_1-2b_2-b_3 \end{array}\right] \quad r_{31}^{(-9)} \quad r_{32}^{(3)}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & -40b_1+16b_2+9b_3 \\ 0 & 1 & 0 & 13b_1-5b_2-3b_3 \\ 0 & 0 & 1 & 5b_1-2b_2-b_3 \end{array}\right]$$

There are no restrictions on b_1, b_2, and b_3

$$\therefore \begin{cases} x_1 = -40b_1+16b_2+9b_3 \\ x_2 = 13b_1-5b_2-3b_3 \\ x_3 = 5b_1-2b_2-b_3 \end{cases} \quad \text{for } b_1, b_2, b_3 \in \mathbb{F}$$