Diagonal, Triangular and Symmetric Matrices

A general $n \times n$ diagonal matrix D:

$$D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix}$$

A diagonal matrix is **invertible** iff all of its diagonal entries are **nonzero.**

$$D^{-1} = \begin{bmatrix} 1/d_1 & & 0 \\ & 1/d_2 & \\ 0 & & \ddots & 1/d_n \end{bmatrix}$$

$$DD^{-1} = D^{-1}D = I_n$$

Also,

$$D^k = \begin{bmatrix} d_1^k & & 0 \\ & d_2^k & \\ 0 & & \ddots & d_n^k \end{bmatrix}$$

<u>Example</u>: If $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 2 \end{bmatrix} \Rightarrow$ diagonal!

then

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}, \quad A^5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -243 & 0 \\ 0 & 0 & 32 \end{bmatrix}$$

$$A^{-5} = (A^5)^{-1} = (A^{-1})^5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{-243} & 0 \\ 0 & 0 & \frac{1}{32} \end{bmatrix}$$

Note that

$$\begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} = \begin{bmatrix} d_1 a_{11} & d_1 a_{12} & d_1 a_{13} & d_1 a_{14} \\ d_2 a_{21} & d_2 a_{22} & d_2 a_{23} & d_2 a_{24} \\ d_3 a_{31} & d_3 a_{32} & d_3 a_{33} & d_3 a_{34} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix} \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} = \begin{bmatrix} d_1 a_{11} & d_2 a_{12} & d_3 a_{13} \\ d_1 a_{21} & d_2 a_{22} & d_3 a_{23} \\ d_1 a_{31} & d_2 a_{32} & d_3 a_{33} \\ d_1 a_{41} & d_2 a_{42} & d_3 a_{43} \end{bmatrix}$$

Upper triangular matrix

$$\begin{bmatrix} a_{11} & & * \\ & a_{22} & \\ & & \ddots \\ 0 & & & a_{nn} \end{bmatrix}$$

Lower triangular matrix

$$\begin{bmatrix} a_{11} & & 0 \\ & a_{22} & \\ & & \ddots \\ * & & & a_{nn} \end{bmatrix}$$

Theorem

(a) The transpose of a lower triangular matrix is upper triangular.
" upper " lower "

(b) The product of lower triangular matrices is lower triangular.
" upper " upper "

(c) A triangular matrix is invertible iff its diagonal entries are all nonzero.

(d) The inverse of an invertible lower triangular matrix is lower triangular.
" upper " upper "

proof (b):

Let $A = [a_{ij}]$, $B = [b_{ij}]$ be two lower triangular matrices

Let $C = [c_{ij}]$ be the product $C = AB$

Assume that $i < j$:

$$c_{ij} = \underbrace{a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{i(j-1)}b_{(j-1)j}}_{\text{row number of } b \,<\, \text{column number of } b} + \underbrace{a_{ij}b_{jj} + \cdots + a_{in}b_{nj}}_{\text{row number of } a \,<\, \text{column number of } a}$$

$= 0$

$\therefore$ for $i<j$, $c_{ij} = 0$ $\quad \begin{bmatrix} c_{11} & & 0 \\ & c_{22} & \\ * & & c_{nn} \end{bmatrix}$

__Example__:

all nonzero

Consider $A = \begin{bmatrix} 1 & 3 & -1 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -2 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

$A^{-1} = \begin{bmatrix} 1 & -\frac{3}{2} & \frac{7}{5} \\ 0 & \frac{1}{2} & -\frac{2}{5} \\ 0 & 0 & \frac{1}{5} \end{bmatrix}$, B is NOT invertible.

$AB = \begin{bmatrix} 3 & -2 & -2 \\ 0 & 0 & 2 \\ 0 & 0 & 5 \end{bmatrix}$, $BA = \begin{bmatrix} 3 & 5 & -1 \\ 0 & 0 & -5 \\ 0 & 0 & 5 \end{bmatrix}$

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & 3 & -1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 & 1 & 0 \\ 0 & 0 & 5 & 0 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 3 & -1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{5} \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & -7 & 1 & -\frac{3}{2} & 0 \\ 0 & 1 & 2 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{5} \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -\frac{3}{2} & \frac{7}{5} \\ 0 & 1 & 0 & 0 & \frac{1}{2} & -\frac{2}{5} \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{5} \end{array}\right]$$

Recall:

A square matrix A is symmetric if $A = A^T$

$\Updownarrow$

$\forall i, j,$

$(A)_{ij} = (A^T)_{ji}$

$= A_{ji}$

* diagonal matrices are symmetric.

Theorem If A and B are symmetric matrices, with the same size, and if $k \in \mathbb{F}$ is any scalar. then

(a) A^T is symmetric $\because A^T = A$, $(A^T)^T = A$ we have proved it before
$= A^T$ from Theorem 1.4.8

(b) $A+B$ and $A-B$ are symmetric. $(A+B)^T = A^T + B^T$

(c) kA is symmetric. $(kA)^T = kA^T = kA$

Note: Product of symmetric matrices is NOT necessarily symmetric!

$$(AB)^T = B^T A^T = BA \overset{?}{=} AB$$

Theorem For two symmetric matrices with the same sizes, say A, B, AB is symmetric iff the matrices commute

i.e; $AB = BA$

Example:

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} -4 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -5 & 2 \end{bmatrix} \Rightarrow \text{NOT symmetric!}$$

$$A \leftarrow \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} -4 & 3 \\ 3 & -1 \end{bmatrix} \Rightarrow B = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \quad AB = BA$$

$$\because \begin{bmatrix} -4 & 3 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} //$$

Invertibility of Symmetric Matrices

Theorem If A is an invertible symmetric matrix, then A^{-1} is symmetric.

(sol): Assume that A is symmetric and invertible

$$\underline{(A^{-1})^T} = (A^T)^{-1} = \underline{A^{-1}}$$

Thus A^{-1} is symmetric ※

Recall: If A is invertible then A^T is also invertible and $(A^T)^{-1} = (A^{-1})^T$

$$\boxed{\begin{aligned} A^T(A^{-1})^T &= (A^{-1}A)^T = I \\ (A^{-1})^T A^T &= (AA^{-1})^T = I \end{aligned}}$$

Recall:

If A is invertible, then A^T is invertible ?!

$$A^T(A^{-1})^T = (A^{-1}A)^T = I^T = I$$

and $$(A^{-1})^T A^T = (AA^{-1})^T = I = I$$

$\therefore$ A^T has an inverse : $(A^{-1})^T$

Theorem If A is an invertible matrix, then AA^T and A^TA are also invertible.

(proof): A is invertible $\Rightarrow$ A^T is invertible

$\because$ products of invertible matrices are also invertible (e.g, elementary matrices)

$\therefore$ AA^T and A^TA are invertible ※

$(AA^T)^T = (A^T)^T A^T = AA^T$

$(A^TA)^T = A^T(A^T)^T = A^T A.$

AA^T and A^TA are symmetric!

<u>Example</u> Let $A = \begin{bmatrix} 1 & -2 & 4 \\ 3 & 0 & -5 \end{bmatrix}$

We have

$$A^TA = \begin{bmatrix} 1 & 3 \\ -2 & 0 \\ 4 & -5 \end{bmatrix}\begin{bmatrix} 1 & -2 & 4 \\ 3 & 0 & -5 \end{bmatrix} = \begin{bmatrix} 10 & -2 & -11 \\ -2 & 4 & -8 \\ -11 & -8 & 41 \end{bmatrix}$$

symmetric!

and

$$AA^T = \begin{bmatrix} 1 & -2 & 4 \\ 3 & 0 & -5 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ -2 & 0 \\ 4 & -5 \end{bmatrix} = \begin{bmatrix} 21 & -17 \\ -17 & 34 \end{bmatrix}$$

also symmetric!

<u>Recall</u>:

- Product of invertible matrices $\Rightarrow$ invertible
- Product of symmetric matrices $\not\Rightarrow$ symmetric
- Product of a matrix and its transpose $\Rightarrow$ symmetric

$$(A)_{m\times n}(A^T)_{n\times m} \rightarrow (AA^T)_{m\times m}$$

$$(A^T)_{n\times m}(A)_{m\times n} \longrightarrow (A^TA)_{n\times n}$$

Let $C = AA^T$, then $C^T = (AA^T)^T = (A^T)^T A^T = AA^T$

Let $C = A^TA$, then $C^T = (A^TA)^T = A^T(A^T)^T = A^TA$