Revisit To Linear Transformations (maps)

△ ordered tuple :

$$(s_1, s_2, \cdots, s_n)$$

if $n=2$:

$\Rightarrow$ ordered pair

if $n=3$

$\Rightarrow$ ordered triple

△ column-vector form:

$$\begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix}$$

△ standard basis vector of $\mathbb{R}^n$:

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \cdots, \quad e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

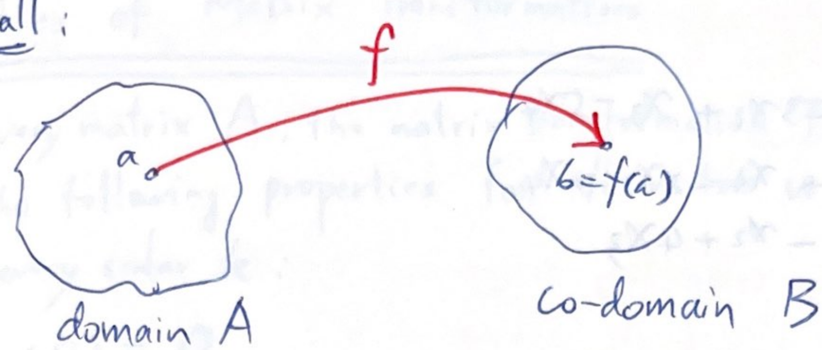
Example The standard basis vector of $\mathbb{R}^3$:

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Any vector in $\mathbb{R}^n$ is expressible as a linear combination of the standard basis vectors.

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{x} = x_1 \cdot e_1 + x_2 e_2 + \cdots + x_n e_n$$

Recall:



b : image of a under f
or the value of f at a

$\{f(a) : a \in A\}$: the range of f

Matrix Transformations

$\Rightarrow$ functions

Suppose we have

$$w_1 = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n$$
$$w_2 = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n$$
$$\vdots$$
$$w_m = a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n$$

We can write it in matrix notation as:

$$\underbrace{\begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{bmatrix}}_{W} = \underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{X}$$

or $W = A \cdot X$

Or, view it as a transformation: (Matrix Transformation)

$$T_A : \mathbb{R}^n \longrightarrow \mathbb{R}^m,$$

$$W = T_A(X)$$

Example:

$$w_1 = 2x_1 - 3x_2 + x_3 - 5x_4$$
$$w_2 = 4x_1 + x_2 - 2x_3 + x_4$$
$$w_3 = 5x_1 - x_2 + 4x_3$$

matrix form $\Longrightarrow$

$$\underbrace{\boxed{\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}}}_{= \, W} = \underbrace{\boxed{\begin{bmatrix} 2 & -3 & 1 & -5 \\ 4 & 1 & -2 & 1 \\ 5 & -1 & 4 & 0 \end{bmatrix}}}_{= \, A} \underbrace{\boxed{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}}_{= \, X}, \quad \text{let } X = \begin{bmatrix} 1 \\ -3 \\ 0 \\ 2 \end{bmatrix}$$

$$T_A(X) = AX = \begin{bmatrix} 2 & -3 & 1 & -5 \\ 4 & 1 & -2 & 1 \\ 5 & -1 & 4 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \\ 0 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 3 \\ 8 \end{bmatrix}$$

Zero Transformation

$0 \in F^{m \times n}$, then

$$T_0(X) = 0 \cdot X = 0$$

Identity Operator

$I \in F^{n \times n}$, then

$$T_I(X) = IX = X$$

Properties of Matrix Transformations

For every matrix A, the matrix transformation $T_A : \mathbb{R}^n \mapsto \mathbb{R}^m$ has the following properties for all vectors u and v and for every scalar k:

(a) $T_A(0) = 0$

(b) $T_A(ku) = kT_A(u)$

(c) $T_A(u+v) = T_A(u) + T_A(v)$

(d) $T_A(u-v) = T_A(u) - T_A(v)$

$\downarrow$ $T_A(k_1u_1 + k_2u_2 + \cdots + k_ru_r)$

$= k_1T_A(u_1) + k_2T_A(u_2) + \cdots + k_rT_A(u_r)$

Theorem: $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ is a matrix transformation iff the following relationships hold for all vectors $u, v \in \mathbb{R}^n$ and for every scalar k:

(i) $T(u+v) = T(u) + T(v)$ linearity conditions

(ii) $T(ku) = kT(u)$

T: linear transformation

Theorem: If $T_A : \mathbb{R}^n \mapsto \mathbb{R}^m$ and $T_B : \mathbb{R}^n \mapsto \mathbb{R}^m$ are matrix transformation and $T_A(\mathbf{x}) = T_B(\mathbf{x})$ for every vector $\mathbf{x}$ in $\mathbb{R}^n$.

Then, $A = B$

(proof):

$T_A(\mathbf{x}) = T_B(\mathbf{x}) \Leftrightarrow A\mathbf{x} = B\mathbf{x}$. Let $\mathbf{x}$ be any standard basis vector,

$Ae_j = Be_j$, $j = 1, 2, \cdots, n$ (Ae_j : the jth column of A)

Note that, for the linear transformation $T: \mathbb{R}^n \mapsto \mathbb{R}^n$

$$A = [T(e_1) | T(e_2) | \cdots | T(e_n)]$$

$\Rightarrow$ A can be completely determined by T's actions on the standard basis vectors of $\mathbb{R}^n$

Example: Find the standard matrix A for the linear transformation $T: \mathbb{R}^2 \mapsto \mathbb{R}^3$ defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 + x_2 \\ x_1 - 3x_2 \\ -x_1 + x_2 \end{bmatrix}$$

sol):
$$T(e_1) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$
$$T(e_2) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix}$$

$\therefore$ the standard matrix is $A = [T(e_1) | T(e_2)]$
$$= \begin{bmatrix} 2 & 1 \\ 1 & -3 \\ -1 & 1 \end{bmatrix}$$

Check!

$$A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & -3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 + x_2 \\ x_1 - 3x_2 \\ -x_1 + x_2 \end{bmatrix}$$

And, how about $T\left(\begin{bmatrix} 1 \\ 4 \end{bmatrix}\right)$?

$$T\left(\begin{bmatrix} 1 \\ 4 \end{bmatrix}\right) = A \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & -3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 6 \\ -11 \\ 3 \end{bmatrix}$$

Example Rewrite $T(x_1, x_2) = (3x_1 + x_2 , 2x_1 - 4x_2)$ in column-vector form and find its standard matrix

(sol):

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 3x_1 + x_2 \\ 2x_1 - 4x_2 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 3 & 1 \\ 2 & -4 \end{bmatrix}}_{\substack{T(e_1) \quad T(e_2) \\ \text{the standard matrix of } T}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Example: Find the standard matrix for the linear transformation $T: \mathbb{R}^2 \longmapsto \mathbb{R}^2$, for which:

$$T\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -5 \\ 5 \end{bmatrix}, \quad T\left(\begin{bmatrix} 2 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 7 \\ -6 \end{bmatrix}$$

rewrite the "standard basis vectors" by them!

(sol):

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix} = k_1 \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} + k_2 \cdot \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$\Rightarrow \begin{cases} c_1 = 1 \\ c_2 = 1 \end{cases}, \quad \begin{cases} k_1 = 2 \\ k_2 = 1 \end{cases}$$

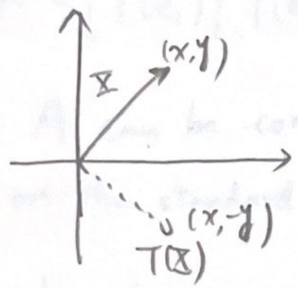
$$\therefore T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = c_1 \cdot T\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right) + c_2 \cdot T\left(\begin{bmatrix} 2 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} -5 \\ 5 \end{bmatrix} + \begin{bmatrix} 7 \\ -6 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = k_1 \cdot T\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right) + k_2 \cdot T\left(\begin{bmatrix} 2 \\ -1 \end{bmatrix}\right) = 2 \cdot \begin{bmatrix} -5 \\ 5 \end{bmatrix} + \begin{bmatrix} 7 \\ -6 \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$$

$$\therefore A = \begin{bmatrix} 2 & -3 \\ -1 & 4 \end{bmatrix}$$

Reflection Operators.

$T(x,y) = (x, -y)$

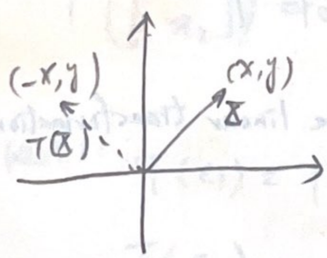


$T(e_1) = T(1,0) = (1,0)$

$T(e_2) = T(0,1) = (0,-1)$

$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$T(x,y) = (-x, y)$

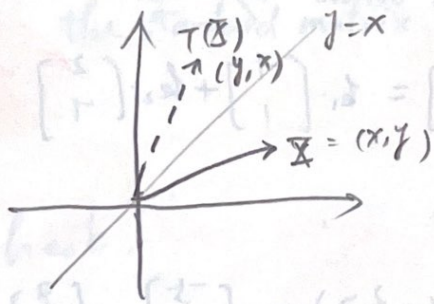


$T(e_1) = (-1,0)$

$T(e_2) = (0,1)$

$\Rightarrow \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

$T(x,y) = (y, x)$



$T(e_1) = (0,1)$

$T(e_2) = (1,0)$

$\Rightarrow \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

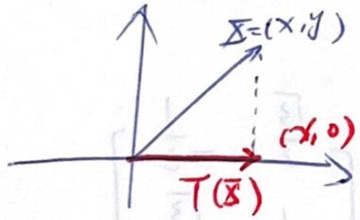
$T(x,y,z) = (x,y,-z)$ $\quad T(e_1) = (1,0,0)$, $T(e_2) = (0,1,0)$, $T(e_3) = (0,0,-1)$ $\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

$T(x,y,z) = (x,-y,z) \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$T(x,y,z) = (-x,y,z) \Rightarrow \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

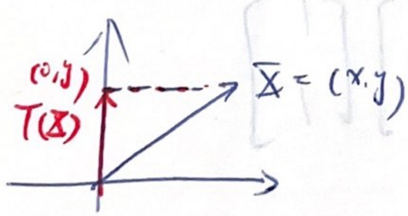
Projection operators :

$T(x,y) = (x,0)$



$T(e_1) = (1,0)$, $T(e_2) = (0,0)$ $\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$T(x,y) = (0,y)$



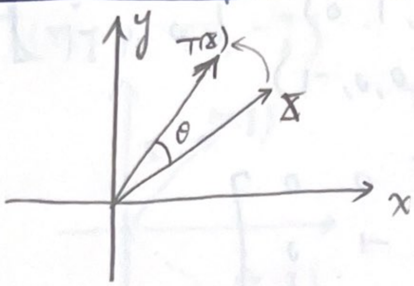
$T(e_1) = (0,0)$, $T(e_2) = (0,1)$ $\Rightarrow \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$\therefore T(x,y,z) = (x,y,0) \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$T(x,y,z) = (x,0,z) \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$T(x,y,z) = (0,y,z) \Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

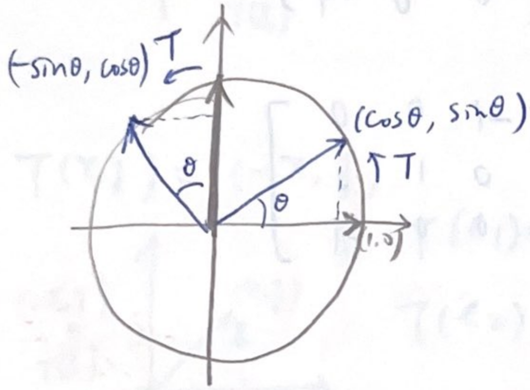
Rotation operators



$T(e_1) = T(1,0) = (\cos\theta, \sin\theta)$

$T(e_2) = T(0,1) = (-\sin\theta, \cos\theta)$

$$\therefore A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$



for θ : counterclockwise

Example :

$\frac{\pi}{6}$

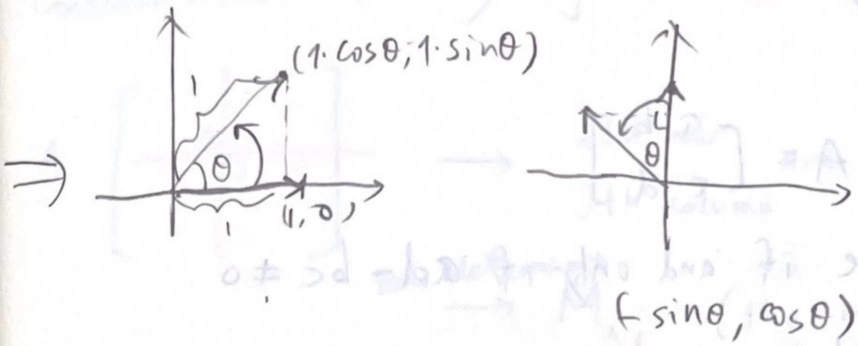
$x = (1,1) \xrightarrow{T:\ \text{rotate } 30^\circ}\ ?$

$$A = \begin{bmatrix} \cos\frac{\pi}{6} & -\sin\frac{\pi}{6} \\ \sin\frac{\pi}{6} & \cos\frac{\pi}{6} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

$$\therefore T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\sqrt{3}-1}{2} \\ \frac{1+\sqrt{3}}{2} \end{bmatrix}$$

$$\Rightarrow \left(\frac{\sqrt{3}-1}{2}, \frac{1+\sqrt{3}}{2}\right)$$



(1·cosθ, 1·sinθ)
1
θ
(1, 0)
1
θ
(-sinθ, cosθ)