Determinants

Recall that for $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

it is invertible if and only if $ad - bc \neq 0$

$\det(A) = ad - bc$

Why?

Consider $\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right]$

$\xrightarrow{r_1^{(\frac{1}{a})}} \left[\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ c & d & 0 & 1 \end{array}\right]$

$\xrightarrow{r_{12}^{(-c)}} \left[\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & \frac{ad-bc}{a} & -\frac{c}{a} & 1 \end{array}\right]$

$\xrightarrow{r_2^{(\frac{a}{ad-bc})}} \left[\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right]$

$\xrightarrow{r_{21}^{(-\frac{b}{a})}} \left[\begin{array}{cc|cc} 1 & 0 & \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right]$

$\therefore$ The inverse exists as long as $ad - bc \neq 0$

<u>Question</u>: How to generalize determinant for $A \in \mathbb{F}^{n \times n}$, $n \geq 2$?

Consider "minor of entry a_{ij}" first!

A: $\begin{bmatrix} & & \\ & a_{ij} & \\ & & \end{bmatrix}$ $\longrightarrow$ ith row, jth column are deleted from A $\rightarrow$ $\xrightarrow{\text{take determinant}}$ M_{ij} (the minor of the entry a_{ij})

☆ $(-1)^{i+j} \cdot M_{ij}$: cofactor of entry a_{ij}

Example:

$$A = \begin{bmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{bmatrix}$$

$$M_{11} = \det\left(\begin{bmatrix} 5 & 6 \\ 4 & 8 \end{bmatrix}\right)$$

$$\text{or } = \begin{vmatrix} 5 & 6 \\ 4 & 8 \end{vmatrix} = 8\times5 - 4\times6 = 16$$

Cofactor of a_{11} is

$$C_{11} = (-1)^{1+1} M_{11} = M_{11} = 16$$

$$M_{32} = \det\left(\begin{bmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{bmatrix}\right)$$

$$= \begin{vmatrix} 3 & -4 \\ 2 & 6 \end{vmatrix} = 18+8 = 26$$

Cofactor of a_{32} is $C_{32} = (-1)^{3+2} \cdot M_{32} = -26$

Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$= a_{11}\overset{a_{22}}{C_{11}} + a_{12}\overset{-a_{21}}{C_{12}} \rightarrow$$

$$= a_{21}\overset{-a_{12}}{C_{21}} + a_{22}\overset{a_{11}}{C_{22}} \rightarrow$$

$$= a_{11}C_{11} + a_{21}C_{21} \rightarrow$$

$$= a_{12}C_{12} + a_{22}C_{22} \rightarrow$$

cofactor expansion

Theorem: For $A \in \mathbb{F}^{n\times n}$, then the cofactor expansion on each row or each column is **the same**.

Definition: For $A \in \mathbb{F}^{n\times n}$,

$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

and

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

for each $i, j \in \{1, 2, \cdots, n\}$

Example:

$$A = \begin{bmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{bmatrix}$$

$$\det(A) = \begin{vmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{vmatrix} = 3\begin{vmatrix} -4 & 3 \\ 4 & -2 \end{vmatrix} - 1\cdot\begin{vmatrix} -2 & 3 \\ 5 & -2 \end{vmatrix} + 0\cdot\begin{vmatrix} -2 & -4 \\ 5 & 4 \end{vmatrix}$$

$$= 3(-4) - 1\cdot(-11) = -1$$

Also,

$$\det(A) = 3 \cdot \begin{vmatrix} -4 & 3 \\ 4 & -2 \end{vmatrix} - (-2) \begin{vmatrix} 1 & 0 \\ 4 & -2 \end{vmatrix} + 5 \begin{vmatrix} 1 & 0 \\ -4 & 3 \end{vmatrix}$$

$$= 3(-4) - (-2)(-2) + 5(3) = -1$$

$\Rightarrow$ Make a smarter choice!

Example: For a lower triangular matrix,

$$\begin{vmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & 0 & 0 \\ a_{32} & a_{33} & 0 \\ a_{42} & a_{43} & a_{44} \end{vmatrix}$$

$$= a_{11} a_{22} \begin{vmatrix} a_{33} & 0 \\ a_{43} & a_{44} \end{vmatrix}$$

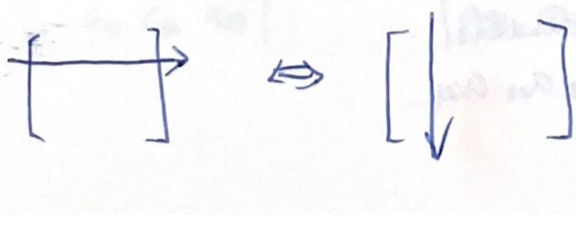
$$= a_{11} a_{22} a_{33} |a_{44}|$$

$$= a_{11} a_{22} a_{33} a_{44}$$ $\Rightarrow$ product of its diagonal entries!

Theorem: If $A \in \mathbb{F}^{n \times n}$ is triangular, then $\det(A) = a_{11} a_{22} \cdots a_{nn}$

Theorem: Let A be a square matrix. If A has any row or column that is full of zeros, then $\det(A) = 0$

Theorem Let A be a square matrix. Then $\det(A) = \det(A^T)$



Theorem 2.2.3 Let $A \in \mathbb{F}^{n\times n}$

(a) If $A \xrightarrow{r_i^{(k)}} B$ or $A \xrightarrow{C_i^{(k)}} B$ for some $i \in [n]$ and scalar k

column

then $\det(B) = k \cdot \det(A)$

(b) If $A \xrightarrow{r_{ij}} B$ or $A \xrightarrow{C_{ij}} B$ for some $i,j \in [n]$

then $\det(B) = -\det(A)$

(c) If $A \xrightarrow{r_{ij}^{(k)}} B$ or $A \xrightarrow{C_{ij}^{(k)}} B$ for some $i,j \in [n]$ and scalar k

then $\det(B) = \det(A)$

Example

$$\begin{vmatrix} ka_{11} & ka_{12} & ka_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = k \cdot \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = - \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\begin{vmatrix} a_{11}+ka_{21} & a_{12}+ka_{22} & a_{13}+ka_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\hookrightarrow (a_{11}+ka_{21}) \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - (a_{12}+ka_{22}) \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + (a_{13}+ka_{23}) \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= \det(A) + \cancel{ka_{21}a_{22}a_{33}} - \cancel{ka_{21}a_{23}a_{32}}$$
$$- \cancel{ka_{22}a_{21}a_{33}} + \cancel{ka_{22}a_{23}a_{31}}$$
$$+ \cancel{ka_{23}a_{21}a_{32}} - \cancel{ka_{23}a_{22}a_{31}}$$

$= \det(A)$

Try to prove (a):

$$\begin{vmatrix} ka_{11} & ka_{12} & ka_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = ka_{11}C_{11} + ka_{12}C_{12} + ka_{13}C_{13}$$

$$= k(a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13})$$

$$= k\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

2.2.4

Theorem Let E be an elementary matrix, $E \in \mathbb{F}^{n\times n}$

(a) If $I_n \xrightarrow{r_i^{(k)}} E$ for some scalar $k \neq 0$ and $i \in [n]$, then $\det(E) = k$.

(b) If $I_n \xrightarrow{r_{ij}} E$ for some $i, j \in [n]$ then $\det(E) = -1$

(c) If $I_n \xrightarrow{r_{ij}^{(k)}} E$ for some scalar k and $i, j \in [n]$, then $\det(E) = 1$

$$\det\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}\right) = -2\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

2.2.5

Theorem If $A \in \mathbb{F}^{n\times n}$ has two proportional rows or two proportional columns, then $\det(A) = 0$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ ka_{11} & ka_{12} & ka_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \xrightarrow{r_{12}^{(-k)}} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ 0 & 0 & 0 \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 0$$

Example

$$\begin{vmatrix} 1 & 3 & -2 & 4 \\ 2 & 6 & -4 & 8 \\ 3 & 9 & 1 & 5 \\ 1 & 1 & 4 & 8 \end{vmatrix} \overset{R_{12}^{(-2)}}{=} \begin{vmatrix} 1 & 3 & -2 & 4 \\ 0 & 0 & 0 & 0 \\ 3 & 9 & 1 & 5 \\ 1 & 1 & 4 & 8 \end{vmatrix} = 0$$

Using row reduction helps !!!

Example: Evaluate $\det(A)$, where $A = \begin{bmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{bmatrix}$

(sol):

$$\det(A) = \begin{vmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{vmatrix}$$

$$= -\begin{vmatrix} 3 & -6 & 9 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix}$$

$$= -3\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix} \; R_{13}^{(-2)}$$

$$= -3\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 10 & -5 \end{vmatrix} \; R_{23}^{(-10)}$$

$$= -3\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & -55 \end{vmatrix}$$

$$= (-3)(-55)\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= (-3)(-55)(1) = 165$$

Example $A = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 2 & 7 & 0 & 6 \\ 0 & 6 & 3 & 0 \\ 7 & 3 & 1 & -5 \end{bmatrix}$, Compute $\det(A)$.

(sol):

$$\det(A) = \begin{vmatrix} 1 & 0 & 0 & 3 \\ 2 & 7 & 0 & 6 \\ 0 & 6 & 3 & 0 \\ 7 & 3 & 1 & -5 \end{vmatrix}$$

$\times(-3)$ or $C_{14}^{(-3)}$

$$= \begin{vmatrix} 1 & 0 & 0 & 0 \\ 2 & 7 & 0 & 0 \\ 0 & 6 & 3 & 0 \\ 7 & 3 & 1 & -26 \end{vmatrix}$$

$$= (1)\cdot(7)\cdot(3)\cdot(-26) = -546$$

Example $A = \begin{bmatrix} 3 & 5 & -2 & 6 \\ 1 & 2 & -1 & 1 \\ 2 & 4 & 1 & 5 \\ 3 & 7 & 5 & 3 \end{bmatrix}$ Evaluate $\det(A)$.

$\times(-2)$, $\times(-3)$

(sol):

$$\det(A) = \begin{vmatrix} 0 & -1 & 1 & 3 \\ 1 & 2 & -1 & 1 \\ 0 & 0 & 3 & 3 \\ 0 & 1 & 8 & 0 \end{vmatrix}$$

$$= (-1)\cdot\begin{vmatrix} -1 & 1 & 3 \\ 0 & 3 & 3 \\ 1 & 8 & 0 \end{vmatrix} \quad r_{13}^{(1)}$$

$$= (-1)\cdot\begin{vmatrix} -1 & 1 & 3 \\ 0 & 3 & 3 \\ 0 & 9 & 3 \end{vmatrix} = (-1)\cdot(-1)\begin{vmatrix} 3 & 3 \\ 9 & 3 \end{vmatrix}$$

$$= -18$$