Properties of Determinants & Cramer's Rule

* Basic properties:

$A, B \in \mathbb{F}^{n\times n}$, k is any scalar

- $\det(kA) = k^n \det(A)$

<u>example</u>: $\begin{vmatrix} ka_{11} & ka_{12} & ka_{13} \\ ka_{21} & ka_{22} & ka_{23} \\ ka_{31} & ka_{32} & ka_{33} \end{vmatrix}$

$= k^3 \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

Note:

$\det(A+B) \neq \det(A) + \det(B)$

<u>Example</u>:

$A = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$, $A+B = \begin{bmatrix} 4 & 3 \\ 3 & 8 \end{bmatrix}$

We have $\det(A) = 1$, $\det(B) = 8$, and $\det(A+B) = 23$

$\therefore \det(A+B) \neq \det(A) + \det(B)$

However, ...

Theorem: Let $A, B, C \in \mathbb{F}^{n\times n}$,

They differ ONLY in a single row, say the rth row,

and $C_{rj} = A_{rj} + B_{rj}$ for $j = 1, 2, \cdots, n$

Then $\det(C) = \det(A) + \det(B)$

Example

$$\det\left(\begin{bmatrix} 1 & 7 & 5 \\ 2 & 0 & 3 \\ 1 & 4 & 7 \end{bmatrix}\right) + \det\left(\begin{bmatrix} 1 & 7 & 5 \\ 2 & 0 & 3 \\ 0 & 1 & -1 \end{bmatrix}\right)$$

$$= \det\left(\begin{bmatrix} 1 & 7 & 5 \\ 2 & 0 & 3 \\ 1+0, & 4+1, & 7+(-1) \end{bmatrix}\right)$$

△ 2.3.2

Lemma Let $B \in \mathbb{F}^{n\times n}$ (不一定要 invertible) and E is an $n\times n$ elementary matrix,

then $\det(EB) = \det(E)\det(B)$

(proof):

Case 1: $E = R_i^{(k)} \Rightarrow \det(E) = k$

$\det(EB) = k\det(B) = \det(E)\cdot\det(B)$

Case 2: $E = R_{ij} \Rightarrow \det(E) = -1$

Case 3: $E = R_{ij}^{(k)} \Rightarrow \det(E) = 1$

$\therefore \det(E_1E_2\cdots E_r B) = \det(E_1)\det(E_2)\cdots\det(E_r)\det(B)$

↳不一定要 invertible!

2.3.3

Theorem: A square matrix A is invertible if and only if $\det(A) \neq 0$

for general $\mathbb{F}^{n\times n}$ matrices!

(proof):

$(\Rightarrow)$: Let R be the reduced row echelon form of A

$\therefore R = E_r E_{r-1} \cdots E_2 \cdot E_1 \cdot A$, where $E_r, E_{r-1}, \cdots, E_1$ are elementary matrices

$\therefore \det(R) = \det(E_r)\det(E_{r-1})\cdots\det(E_2)\det(E_1)\det(A)$

note that $\det(E_i) \neq 0$ for $i = 1, 2, \cdots, r$

could only be $1, -1, k>0$

$(\Rightarrow)$

$\because A$ is invertible

$\therefore R = I_n \Rightarrow \det(R) = 1$

$\Rightarrow \det(A) \neq 0$

$(\Leftarrow)$: $\because \det(A) \neq 0$

$\therefore \det(E_r)\cdot\det(E_{r-1})\cdot \cdots \cdot \det(E_2)\cdot\det(E_1)\cdot\det(A) \neq 0$

$\Rightarrow \det(R) \neq 0$

$\Rightarrow R$ does not have a row of zeros

$\Rightarrow R = I_n$

$\because R$ is the RREF

Example:
$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 2 & 4 & 6 \end{bmatrix} \quad r_{13}^{(-2)}$$

$$\det(A) = \begin{vmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{vmatrix} = 0 \Rightarrow A \text{ is NOT invertible}$$

2.3.4

Theorem If $A, B \in \mathbb{F}^{n\times n}$ (→square!), then $\det(AB) = \det(A)\cdot\det(B)$

(NO MATTER A and B are invertible or NOT)

(proof):

If A or B $= O$, it's trivial. So, assume that $A \neq O$, $B \neq O$

Case 1: A is NOT invertible

$\Rightarrow$ AB is NOT invertible

why? Assume the contrary that AB is invertible.

Let X_0 be any solution of $BX = 0$

$\Rightarrow (AB)X_0 = A(BX_0) = AO = 0$

$\because$ AB is invertible

$\therefore X_0$ must be O (trivial solution)

$\Rightarrow BX = 0$ only has the trivial solution

$\Rightarrow$ B is invertible

$\Rightarrow A = (AB)\cdot B^{-1}$ is the product of invertible matrices

$\Rightarrow$ A is invertible ($\Rightarrow\Leftarrow$)

$\therefore \det(AB) = 0 = \det(A)$

$\Rightarrow \det(AB) = \det(A)\cdot\det(B)$

Case 2: A is invertible

$\Rightarrow A = E_1 E_2 \cdots E_r$ for some integer r

$\therefore AB = E_1 E_2 \cdots E_r B$

$\therefore \det(AB) = \underbrace{\det(E_1)\det(E_2)\cdots\det(E_r)}\det(B)$

$= \det(E_1\cdot E_2\cdots E_r)\det(B)$

$= \det(A)\cdot\det(B)$

2.3.5

Theorem: If A is invertible, then $\det(A^{-1}) = \frac{1}{\det(A)}$

(proof): Since $A^{-1}A = I$

$\Rightarrow \det(A^{-1}A) = \det(I) = 1$

$\Rightarrow \det(A^{-1})\det(A) = 1$

$\therefore \det(A^{-1}) = \frac{1}{\det(A)}$ ✠

Adjoint of a Matrix

Observation: multiply entries in any row by the corresponding cofactors from a **different** row

Example:

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{bmatrix}$$

$C_{11} = 12 \quad C_{12} = 6 \quad C_{13} = -16$

$C_{21} = 4 \quad C_{22} = 2 \quad C_{23} = 16$

$C_{31} = 12 \quad C_{32} = -10 \quad C_{33} = 16$

$\therefore \det(A) = 3C_{11} + 2C_{12} + (-1)C_{13} = 36 + 12 + 16 = 64$

$\det(A) = 3C_{11} + 1 \cdot C_{21} + 2 \cdot C_{31} = 36 + 4 + 24 = 64$

But, $3C_{21} + 2C_{22} + (-1)C_{23} = 12 + 4 + (-16) = 0$

and $3C_{12} + 1 \cdot C_{22} + 2C_{32} = 18 + 2 - 20 = 0$

Definition Matrix of Cofactors from A.

$$M_c = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}$$

where C_{ij} is the cofactor of entry a_{ij}

We denote by $adj(A) := M_c^T$

Example:

Let $A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{bmatrix}$

We have $C_{11} = 12$ $C_{12} = 6$ $C_{13} = -16$

$C_{21} = 4$ $C_{22} = 2$ $C_{23} = 16$

$C_{31} = 12$ $C_{32} = -10$ $C_{33} = 16$

$$\therefore M_c = \begin{bmatrix} 12 & 6 & -16 \\ 4 & 2 & 16 \\ 12 & -10 & 16 \end{bmatrix}$$

$$\text{So } adj(A) = \begin{bmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{bmatrix}$$

2.3.6

Theorem If A is an invertible matrix, then

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)$$

(proof): Try to show that $A \cdot \operatorname{adj}(A) = \det(A) \cdot I$

Consider the product:

$$A \operatorname{adj}(A) = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \cdot \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{j1} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{j2} & & C_{n2} \\ \vdots & \vdots & & \vdots & & \vdots \\ C_{1n} & C_{2n} & & C_{jn} & \cdots & C_{nn} \end{bmatrix}$$

The marked product of $A \cdot \operatorname{adj}(A)$:

$$a_{i1} C_{j1} + a_{i2} C_{j2} + \cdots + a_{in} C_{jn} \quad \cdots\cdots (*)$$

① if $i = j$, $(*)$ = cofactor expansion of $\det(A)$
$= \det(A)$

② if $i \neq j$, $(*) = 0$

$$\therefore A \operatorname{adj}(A) = \begin{bmatrix} \det(A) & & & O \\ & \det(A) & & \\ & & \ddots & \\ O & & & \det(A) \end{bmatrix} \quad \cdots\cdots (**)$$

$$= \det(A) \cdot I$$

$\because$ A is invertible, $\Rightarrow \det(A) \neq 0$

$$\therefore (**) \Rightarrow \frac{1}{\det(A)} [A \operatorname{adj}(A)] = I \Rightarrow A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)$$

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Example: In the previous example,

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)$$

$$= \frac{1}{64}\begin{bmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{bmatrix} = \begin{bmatrix} \frac{12}{64} & \frac{4}{64} & \frac{12}{64} \\ \frac{6}{64} & \frac{2}{64} & -\frac{10}{64} \\ -\frac{16}{64} & \frac{16}{64} & \frac{16}{64} \end{bmatrix}$$ ☆

2.3.7

Theorem (Cramer's Rule)

If $A\mathbf{x} = b$ is a system of n linear equations of n unknowns such that $\det(A) \neq 0$, then the system has a unique solution:

$$x_1 = \frac{\det(A_1)}{\det(A)}, \quad x_2 = \frac{\det(A_2)}{\det(A)}, \quad \dots, \quad x_n = \frac{\det(A_n)}{\det(A)}$$

where A_j is the matrix obtained by relacing the entries in the jth column of A by $b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$

Example:

$$\begin{aligned} x_1 + \quad\; 2x_3 &= 6 \\ -3x_1 + 4x_2 + 6x_3 &= 30 \\ -x_1 - 2x_2 + 3x_3 &= 8 \end{aligned}$$

(sol): $A = \begin{bmatrix} 1 & 0 & 2 \\ -3 & 4 & 6 \\ -1 & -2 & 3 \end{bmatrix}$, $b = \begin{bmatrix} 6 \\ 30 \\ 8 \end{bmatrix}$

$\therefore A_1 = \begin{bmatrix} 6 & 0 & 2 \\ 30 & 4 & 6 \\ 8 & -2 & 3 \end{bmatrix}$, $A_2 = \begin{bmatrix} 1 & 6 & 2 \\ -3 & 30 & 6 \\ -1 & 8 & 3 \end{bmatrix}$

$A_3 = \begin{bmatrix} 1 & 0 & 6 \\ -3 & 4 & 30 \\ -1 & -2 & 8 \end{bmatrix}$

$\therefore x_1 = \frac{\det(A_1)}{\det(A)} = \frac{-40}{44} = \frac{-10}{11}$, $x_3 = \frac{\det(A_3)}{\det(A)} = \frac{152}{44} = \frac{38}{11}$

$x_2 = \frac{\det(A_2)}{\det(A)} = \frac{72}{44} = \frac{18}{11}$, #

(proof of Cramer's Rule):

$\det(A) \neq 0 \Rightarrow A$ is invertible

and $X = A^{-1}b$ is the unique solution of $AX = b$.

Using $\mathrm{adj}(A)$ to find A^{-1}, we have

$$X = A^{-1}b = \frac{1}{\det(A)} \mathrm{adj}(A) \cdot b$$

$$= \frac{1}{\det(A)} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$\therefore X = \frac{1}{\det(A)} \begin{bmatrix} b_1 C_{11} + b_2 \cdot C_{21} + \cdots + b_n \cdot C_{n1} \\ b_1 C_{12} + b_2 C_{22} + \cdots + b_n \cdot C_{n2} \\ \vdots \\ b_1 C_{1n} + b_2 \cdot C_{2n} + \cdots + b_n C_{nn} \end{bmatrix}$$

The entry in the jth row of X:

$$x_j = \frac{b_1 C_{1j} + b_2 C_{2j} + \cdots + b_n C_{nj}}{\det(A)} \cdots\cdots (*)$$

Let
$$A_j = \begin{bmatrix} a_{11} & a_{12} \cdots & a_{1j-1} & b_1 & a_{1j+1} & \cdots & a_{1n} \\ a_{21} & a_{22} \cdots & a_{2j-1} & b_2 & a_{2j+1} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} \cdots & a_{nj-1} & b_n & a_{nj+1} & \cdots & a_{nn} \end{bmatrix}$$

$$\det(A_j) = b_1 C_{1j} + b_2 C_{2j} + \cdots + b_n C_{nj}$$

跟原來 A 的 Cofactors 一樣!

That is,

$$x_j = \frac{\det(A_j)}{\det(A)}, \quad \text{for each } j.$$

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