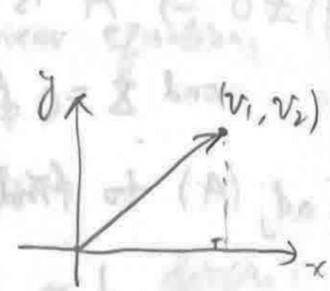
Norm, Dot Product, and Distance in $\mathbb{R}^n$

△ Length of a vector v : $\|v\|$

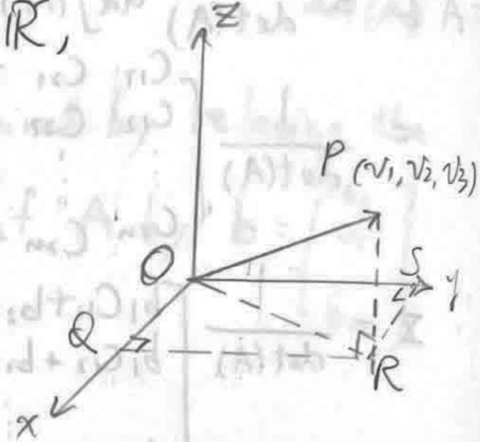
For $v = (v_1, v_2) \in \mathbb{R}^2$,

$$\|v\| = \sqrt{v_1^2 + v_2^2}$$



For $v = (v_1, v_2, v_3) \in \mathbb{R}^3$,

$$\|v\| = \sqrt{v_1^2 + v_2^2 + v_3^2}$$



$$\|v\|^2 = |\overrightarrow{OR}|^2 + |\overrightarrow{RP}|^2$$

$$= |\overrightarrow{OQ}|^2 + |\overrightarrow{QR}|^2 + |\overrightarrow{RP}|^2$$

$$= v_1^2 + v_2^2 + v_3^2$$

Definition (norm).

If $v = (v_1, v_2, \cdots, v_n)$ is a vector in $\mathbb{R}^n$, then the norm of v is denoted by $\|v\|$ and

or magnitude, length

$$\|v\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

Theorem 3.2.1 If v is a vector in $\mathbb{R}^n$ and k is any scalar, then

(a) $\|v\| \geq 0$

(b) $\|v\| = 0$ if and only if $v = \mathbf{0}$

(c) $\|kv\| = |k| \cdot \|v\|$

proof of (c):

If $v = (v_1, v_2, \cdots, v_n)$ then $kv = (kv_1, kv_2, \cdots, kv_n)$

$$\therefore \|kv\| = \sqrt{(kv_1)^2 + (kv_2)^2 + \cdots + (kv_n)^2}$$
$$= \sqrt{k^2(v_1^2 + v_2^2 + \cdots + v_n^2)}$$
$$= |k| \cdot \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$
$$= |k| \cdot \|v\| \quad \#$$

Two vectors in $\mathbb{R}^n$ v_1, v_2

- have the same direction: $v_1 = kv_2,\ k > 0$
- have the opposite direction: $v_1 = kv_2,\ k < 0$

<u>unit vector</u>: a vector of norm 1.

If $v \neq \mathbf{0}$ is any vector in $\mathbb{R}^n$,
then $u = \frac{1}{\|v\|} v$
is a unit vector in the same direction as v.

Example Find the unit vector u that has the same direction as $v = (2, 2, -1)$

(Sol): $\|v\| = \sqrt{2^2 + 2^2 + (-1)^2} = 3$

$$\therefore u = \frac{1}{3}(2, 2, -1) = \left(\frac{2}{3}, \frac{2}{3}, \frac{-1}{3}\right) \quad \#$$

The standard unit vectors

in $\mathbb{R}^n$:

$$e_1 = (1, 0, 0, \cdots, 0)$$
$$e_2 = (0, 1, 0, \cdots, 0)$$
$$\vdots$$
$$e_n = (0, 0, \cdots, 0, 1)$$

Definition (distance in $\mathbb{R}^n$)

If $u = (u_1, u_2, \cdots, u_n)$ and $v = (v_1, v_2, \cdots, v_n)$ are two points in $\mathbb{R}^n$, then $d(u, v)$ is the distance between u and v

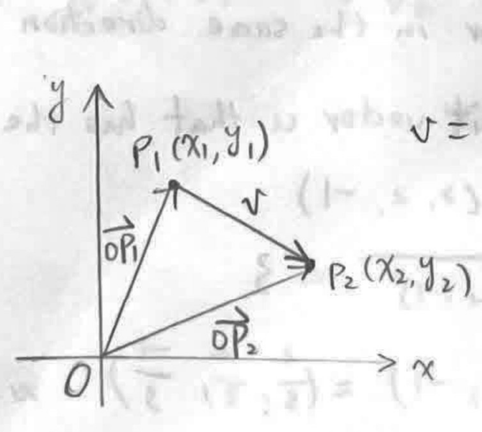
$$d(u,v) = \|u - v\| = \sqrt{(u_1-v_1)^2 + (u_2-v_2)^2 + \cdots + (u_n-v_n)^2}$$

Example $u = (1, 3, -2, 7)$, $v = (0, 7, 2, 2)$

$$d(u,v) = \sqrt{(1-0)^2 + (3-7)^2 + (-2-2)^2 + (7-2)^2}$$

$$= \sqrt{58}$$

Note :



$$v = \overrightarrow{P_1P_2} = \overrightarrow{OP_2} - \overrightarrow{OP_1}$$
$$= (x_2 - x_1,\ y_2 - y_1)$$

Dot product

If u and v are nonzero vectors in R^2 or R^3, and θ is the angle between u and v, then the dot product (Euclidean inner product) of u and v is

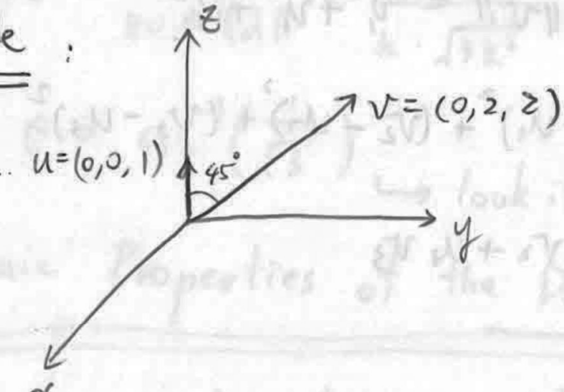
$$u \cdot v = \|u\| \cdot \|v\| \cdot \cos\theta$$

If $u = 0$ or $v = 0$, then $u \cdot v := 0$

Note : ① $0 \le \theta \le \pi$

② $\cos\theta = \dfrac{u \cdot v}{\|u\| \|v\|}$

Example :

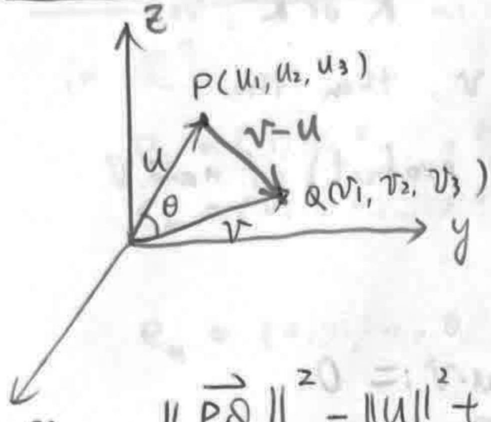


$\|u\| = 1, \quad \|v\| = \sqrt{8} = 2\sqrt{2}$

$\cos\theta = \cos 45^\circ = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

$\therefore u \cdot v = \|u\| \cdot \|v\| . \cos\theta = 1 \cdot 2\sqrt{2} \cdot \frac{\sqrt{2}}{2} = 2$

Law of Cosines



$$\|\overrightarrow{PQ}\|^2 = \|u\|^2 + \|v\|^2 - 2\|u\|\cdot\|v\|\cdot\cos\theta$$

$$\overrightarrow{PQ} = v-u$$

$$\therefore \boxed{\|u\|\|v\|\cos\theta} = \frac{1}{2}\left(\|u\|^2 + \|v\|^2 - \|v-u\|^2\right)$$

$$\|u\|^2 = u_1^2 + u_2^2 + u_3^2, \quad \|v\|^2 = v_1^2 + v_2^2 + v_3^2$$

and $\|v-u\|^2 = (v_1-u_1)^2 + (v_2-u_2)^2 + (v_3-u_3)^2$

$$\therefore u\cdot v = u_1v_1 + u_2v_2 + u_3v_3$$

???

Definition 4 (The Dot Product)

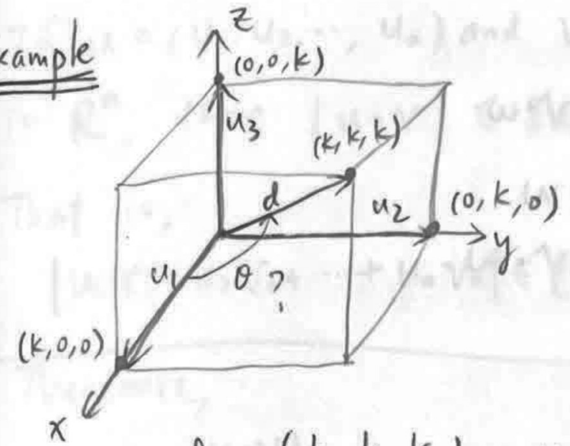
If $u=(u_1, u_2, \cdots, u_n)$ and $v=(v_1, v_2, \cdots, v_n)$ are vectors in $\mathbb{R}^n$, then the dot product (i.e., Euclidean inner product) of u and v is

$$u\cdot v = u_1v_1 + u_2v_2 + \cdots + u_nv_n$$

Example : $u = (-1, 3, 5, 7)$, $v = (-3, -4, 1, 0)$

$u \cdot v = (-1)(-3) + 3 \cdot (-4) + 5 \cdot 1 + 7 \cdot 0 = -4$

Example



$$d = (k, k, k) = u_1 + u_2 + u_3$$

$$\cos\theta = \frac{u_1 \cdot d}{\|u_1\| \cdot \|d\|} = \frac{k^2}{k \cdot \sqrt{3k^2}} = \frac{1}{\sqrt{3}}$$

$$\therefore \theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$$ ↳ look it up ! ⇒ $\theta \simeq 54.74°$

Algebraic Properties of the Dot Product

$$v \cdot v = v_1^2 + v_2^2 + \cdots + v_n^2 = \|v\|^2$$

☆ $\therefore \|v\| = \sqrt{v \cdot v}$

Theorem If u, v, w are vectors in $\mathbb{R}^n$ and k is a scalar.

Then (a) $u \cdot v = v \cdot u$ (symmetric)

(b) $u \cdot (v + w) = u \cdot v + u \cdot w$ (distributive)

(c) $k(u \cdot v) = (ku) \cdot v$ (homogeneity)

(d) $v \cdot v \geq 0$ and $v \cdot v = 0$ if and only if $v = 0$ (positive)

3.2.3

Theorem If u, v, w are vectors in $\mathbb{R}^n$, and if k is a scalar

Then :

(a) $\mathbf{0} \cdot v = v \cdot \mathbf{0} = 0$

(b) $(u+v) \cdot w = u \cdot w + v \cdot w$

(c) $u \cdot (v - w) = u \cdot v - u \cdot w$

(d) $(u - v) \cdot w = u \cdot w - v \cdot w$

(e) $k(u \cdot v) = u \cdot (kv)$

proof of (b):

$$
\begin{aligned}
(u+v) \cdot w &= w \cdot (u+v) && \text{(symmetric)} \\
&= w \cdot u + w \cdot v && \text{(distributive)} \\
&= u \cdot w + v \cdot w && \text{(symmetric)}
\end{aligned}
$$

Example

$$
\begin{aligned}
&(u - 2v) \cdot (3u + 4v) \\
&= u \cdot (3u + 4v) - 2v \cdot (3u + 4v) \\
&= 3(u \cdot u) + 4(u \cdot v) - 6(v \cdot u) - 8(v \cdot v) \\
&= 3\|u\|^2 - 2(u \cdot v) - 8\|v\|^2
\end{aligned}
$$

Remark:

$$\theta = \cos^{-1}\left(\frac{u \cdot v}{\|u\| \cdot \|v\|}\right)$$

$\therefore \dfrac{u \cdot v}{\|u\| \, \|v\|} \in [-1, 1]$ for this formula to be valid!

Fortunately, we have the following theorem!

Theorem (Cauchy - Schwarz Inequality)

If $u = (u_1, u_2, \cdots, u_n)$ and $v = (v_1, v_2, \cdots, v_n)$ are vectors in $\mathbb{R}^n$, then $|u \cdot v| \le \|u\| \cdot \|v\|$

That is,

$$|u_1 v_1 + u_2 v_2 + \cdots + u_n v_n| \le (u_1^2 + u_2^2 + \cdots + u_n^2)^{\frac{1}{2}} (v_1^2 + v_2^2 + \cdots + v_n^2)^{\frac{1}{2}}$$

Therefore,

$$\frac{|u \cdot v|}{\|u\| \, \|v\|} \le 1$$

Geometry in $\mathbb{R}^n$

3.2.5 **Theorem** If u, v, w are vectors in $\mathbb{R}^n$, then

(a) $\|u + v\| \le \|u\| + \|v\|$ (triangular inequality)

(b) $d(u, v) \le d(u, w) + d(w, v)$

(proof)

(a)
$$\begin{aligned}
\|u + v\|^2 &= (u + v) \cdot (u + v) = u \cdot u + 2(u \cdot v) + v \cdot v \\
&= \|u\|^2 + 2(u \cdot v) + \|v\|^2 \\
&\le \|u\|^2 + 2|u \cdot v| + \|v\|^2 \\
&\le \|u\|^2 + 2\|u\| \cdot \|v\| + \|v\|^2 \quad \text{(Cauchy-Schwarz)} \\
&= (\|u\| + \|v\|)^2 \quad \star
\end{aligned}$$

(b)
$$\begin{aligned}
d(u, v) = \|u - v\| &= \|(u - w) + (w - v)\| \\
&\le \|u - w\| + \|w - v\| \\
&= d(u, w) + d(w, v) \quad \star
\end{aligned}$$

Cauchy-Schwarz Inequality
If $u = (u_1, u_2, \cdots, u_n)$ and $v = (v_1, v_2, \cdots, v_n)$ are vectors in $\mathbb{R}^n$, then $|u \cdot v| \leq \|u\| \|v\|$

(proof):
Consider $u \neq 0$ (otherwise it's trivially true)

Let $a = \langle u, u \rangle$, $b = 2\langle u, v \rangle$, $c = \langle v, v \rangle$
hence $a > 0$, $c \geq 0$
Let $t \in \mathbb{R}$ be a scalar
Since $\langle t \cdot u + v, tu + v \rangle \geq 0$
$\Rightarrow t^2 \langle u, u \rangle + 2\langle u, v \rangle t + \langle v, v \rangle \geq 0$
$\Rightarrow at^2 + bt + c \geq 0 \cdots\cdots (*)$
If we consider $f(t) := at^2 + bt + c$
then $f(t) = 0$: $\begin{cases} \text{has no real roots} \\ \text{or has a repeated real root} \end{cases}$

$\Rightarrow$ the discriminant
$$b^2 - 4ac \leq 0$$
$\Rightarrow 4\langle u, v \rangle^2 - 4\langle u, u \rangle \cdot \langle v, v \rangle \leq 0$
$\Rightarrow \langle u, v \rangle^2 \leq \langle u, u \rangle \cdot \langle v, v \rangle$

<u>Theorem</u> If u and v are vectors in $\mathbb{R}^n$ then

① $\|u+v\|^2 + \|u-v\|^2 = 2(\|u\|^2 + \|v\|^2)$

② $u \cdot v = \frac{1}{4}\|u+v\|^2 - \frac{1}{4}\|u-v\|^2$

(proof):

① $\|u+v\|^2 = (u+v)\cdot(u+v) = \|u\|^2 + \|v\|^2 + 2(u\cdot v)$ ...(i)

$\|u-v\|^2 = (u-v)\cdot(u-v) = \|u\|^2 + \|v\|^2 - 2(u\cdot v)$ ...(ii)

② (i) − (ii)

Dot Products as Matrix Multiplication

$A : \mathbb{F}^{n\times n}$, $u, v : \mathbb{F}^{n\times 1}$

$$\boxed{\overset{n\times 1}{Au} \cdot \overset{n\times 1}{v}} = v^T(Au) = (v^T A)u = (\overset{n\times 1}{A^T v})^T \overset{n\times 1}{u} = \boxed{u \cdot A^T v}$$

$$\boxed{u \cdot Av} = (Av)^T u = (v^T A^T)u = \underset{n\times 1}{v}^T (\overset{n\times 1}{A^T u}) = \boxed{A^T u \cdot v}$$

① u : column matrix
v : column matrix

$$\underline{u \cdot v = u^T v = v^T u}$$

$$u = \begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix}, \quad v = \begin{bmatrix} 5 \\ 4 \\ 0 \end{bmatrix}$$

$$u^T v = \begin{bmatrix} 1 & -3 & 5 \end{bmatrix} \begin{bmatrix} 5 \\ 4 \\ 0 \end{bmatrix}$$

$$v^T u = \begin{bmatrix} 5 & 4 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix}$$

② u : row matrix
v : column matrix

$$\underline{u \cdot v = uv = v^T u^T}$$

$$u = \begin{bmatrix} 1 & -3 & 5 \end{bmatrix}, \quad v = \begin{bmatrix} 5 \\ 4 \\ 0 \end{bmatrix}$$

③ u : column matrix
v : row matrix

$$\underline{u \cdot v = vu = u^T v^T}$$

$$u = \begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix}, \quad v = \begin{bmatrix} 5 & 4 & 0 \end{bmatrix}$$

④ u : row matrix
v : row matrix

$$\underline{u \cdot v = uv^T = vu^T}$$

$$u = \begin{bmatrix} 1 & -3 & 5 \end{bmatrix}, \quad v = \begin{bmatrix} 5 & 4 & 0 \end{bmatrix}$$

Example $A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 4 & 1 \\ -1 & 0 & 1 \end{bmatrix}$, $u = \begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix}$, $v = \begin{bmatrix} -2 \\ 0 \\ 5 \end{bmatrix}$

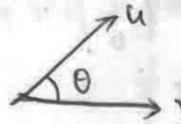
$$Au = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 4 & 1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 7 \\ 10 \\ 5 \end{bmatrix}$$

$$A^T v = \begin{bmatrix} 1 & 2 & -1 \\ -2 & 4 & 0 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 0 \\ 5 \end{bmatrix} = \begin{bmatrix} -7 \\ 4 \\ -1 \end{bmatrix}$$

$\therefore Au \cdot v = 7(-2) + 10(0) + 5(5) = 11$, $\quad u \cdot A^T v = (-1)(-7) + 2(4) + 4(-1) = 11$

Orthogonality

Recall that for two nonzero vectors u, v in $\mathbb{R}^n$



$$\theta = \cos^{-1}\left(\frac{u \cdot v}{\|u\|\,\|v\|}\right)$$

Thus, $\theta = \frac{\pi}{2} = 90^\circ$ if and only if $u \cdot v = 0$

Definition (orthogonal)

① Two nonzero vectors u and v in $\mathbb{R}^n$ are orthogonal if $u \cdot v = 0$.

② The zero vector in $\mathbb{R}^n$ is orthogonal to every vector in $\mathbb{R}^n$

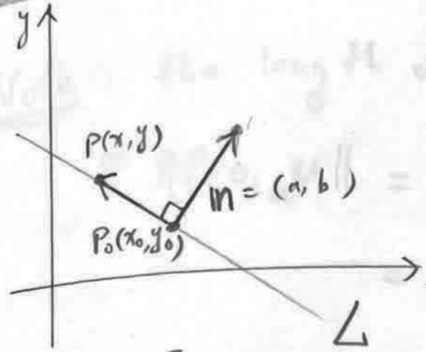
③ A nonempty set of vectors in $\mathbb{R}^n$ is an orthogonal set if "all pairs" of distinct vectors in the set are orthogonal.

④ An orthogonal set of unit vectors is called an orthonormal set.

Example: $u = (-2, 3, 1, 4)$ and $v = (1, 2, 0, -1)$ are orthogonal vectors.

$$u \cdot v = (-2)\cdot 1 + 3\cdot 2 + 1\cdot 0 + 4(-1) = 0$$

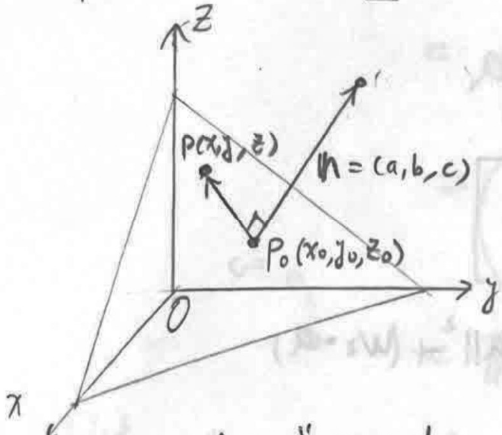
※ **Lines and Planes Determined by Points and Norms.** (點法式)



L:

$$n \cdot \overrightarrow{P_0P} = 0$$

$$\Rightarrow a\cdot(x-x_0) + b(y-y_0) = 0$$



Plane:

$$n \cdot \overrightarrow{P_0P} = 0$$

$$\Rightarrow a\cdot(x-x_0) + b(y-y_0) + c(z-z_0) = 0$$

The "point-normal" equations!
(點法式)

3.3.1
Theorem

(a) If a and b are nonzero constants, then

$$ax + by + c = 0$$

represents a line in $\mathbb{R}^2$ with normal $n = (a,b)$

(b) If a, b, and c are nonzero constants, then

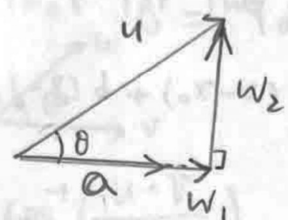
$$ax + by + cz + d = 0$$

represents a plane in $\mathbb{R}^3$ with normal $n = (a,b,c)$

Remark:
for (x,y) on the line $ax+by = 0$

$\because n = (a,b)$ $\therefore$ $(a,b)\cdot(x,y) = ax+by = 0$

Projection Theorem



$u = w_1 + w_2$ → orthogonal to a (for w_2)

w_1: a scalar multiple of a

Say $u = w_1 + w_2 = ka + w_2$

$\therefore u \cdot a = (ka + w_2)\cdot a = k\|a\|^2 + \underbrace{(w_2 \cdot a)}_{=0}$

$\therefore k = \dfrac{u\cdot a}{\|a\|^2}$ and $w_2 = u - w_1$

$= u - ka$

$= u - \dfrac{u\cdot a}{\|a\|^2}\cdot a$

$\text{proj}_a u = \dfrac{u\cdot a}{\|a\|^2}\cdot a$

$u - \text{proj}_a u$

Example $u = (2,-1,3)$ and $a = (4,-1,2)$.

$u\cdot a = 2(4) + (-1)(-1) + 3(2) = 15$

$\|a\|^2 = \sqrt{4^2 + (-1)^2 + 2^2} = 21$

$\text{Proj}_a u = \dfrac{u\cdot a}{\|a\|^2}\cdot a = \dfrac{15}{21}(4,-1,2)$

$= \left(\dfrac{20}{7}, \dfrac{-5}{7}, \dfrac{10}{7}\right)$

and

$u - \text{proj}_a u = (2,-1,3) - \left(\dfrac{20}{7}, \dfrac{-5}{7}, \dfrac{10}{7}\right)$

$= \left(-\dfrac{6}{7}, -\dfrac{2}{7}, \dfrac{11}{7}\right)$

<u>Note</u> : the length of the projection

$$= \|\text{Proj}_{\mathbf{a}}\mathbf{u}\| = \left\|\frac{\mathbf{u}\cdot\mathbf{a}}{\|\mathbf{a}\|^2}\cdot\mathbf{a}\right\|$$

$$= \left|\frac{\mathbf{u}\cdot\mathbf{a}}{\|\mathbf{a}\|^2}\right|\cdot\|\mathbf{a}\|$$

$$= \frac{|\mathbf{u}\cdot\mathbf{a}|}{\|\mathbf{a}\|^2}\cdot\|\mathbf{a}\|$$

$$= \frac{|\mathbf{u}\cdot\mathbf{a}|}{\|\mathbf{a}\|}$$

$$= \|\mathbf{u}\||\cos\theta|$$

<u>Theorem</u> 3.3.3 (Pythagoras)

If u and v are orthogonal vectors in $\mathbb{R}^n$ with the Euclidean inner product, then

$$\|u+v\|^2 = \|u\|^2 + \|v\|^2$$

(proof): $\because$ u and v are orthogonal $\therefore$ $u\cdot v = 0$

$$\|u+v\|^2 = (u+v)\cdot(u+v)$$
$$= \|u\|^2 + 2(u\cdot v) + \|v\|^2 = \|u\|^2 + \|v\|^2$$ #

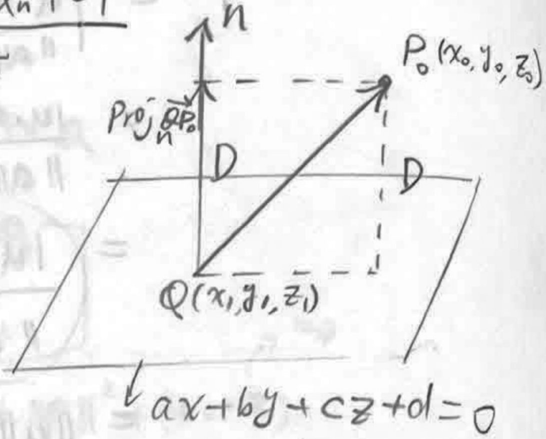
3.3.4

Theorem (distances)

I $\mathbb{R}^n$, the distance D between $P_0(\alpha_1, \alpha_2, \cdots, \alpha_n)$ and the line $a_1x_1 + a_2x_2 + \cdots + a_nx_n + b = 0$

is $D = \dfrac{|a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n + b|}{\sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}}$

(proof of case $\mathbb{R}^3$):



$D = \| proj_n \overrightarrow{QP_0} \|$

$= \dfrac{|\overrightarrow{QP_0} \cdot n|}{\|n\|}$

$\overrightarrow{QP_0} = (x_0 - x_1,\ y_0 - y_1,\ z_0 - z_1)$

$\overrightarrow{QP_0} \cdot n = a(x_0 - x_1) + b(y_0 - y_1) + c(z_0 - z_1)$

and $\|n\| = \sqrt{a^2 + b^2 + c^2}$

$\therefore D = \dfrac{|a(x_0 - x_1) + b(y_0 - y_1) + c(z_0 - z_1)|}{\sqrt{a^2 + b^2 + c^2}}$

$\because$ Q lies in the plane 代入

$\therefore ax_1 + by_1 + cz_1 + d = 0$

$\Rightarrow d = -ax_1 - by_1 - cz_1$

Example (assignment)

Find the distance D between $(1, -4, -3)$ and the plane $2x - 3y + 6z = -1$

Ans: $\frac{3}{7}$