§ General Vector Spaces

Definition [Vector space]

Let $V \neq \emptyset$ with two operations $\begin{cases} \text{addition} & u+v \\ \text{multiplication by scalars} \end{cases}$

$k \cdot v$
$k \in \mathbb{R}$,
$v \in V$

If the following axioms are satisfied by all $u, v, w \in V$ and all scalars k, m, then we call V a vector space.

1. For $u, v \in V$, $u+v \in V$
2. For $u, v \in V$, $u+v = v+u$
3. $u+(v+w) = (u+v)+w$, for $u, v, w \in V$
4. There exists a "zero vector", denoted by $\mathbf{0}$, such that $\mathbf{0}+u = u+\mathbf{0} = u$ for all $u \in V$
5. For each $u \in V$, $-u \in V$ (negative of u), such that $u+(-u) = (-u)+u = \mathbf{0}$
6. For any scalar k and any $u \in V$, $ku \in V$
7. For any scalar k, $k(u+v) = ku+kv$
8. For any two scalars k and m, $(k+m)u = ku+mu$
9. For any two scalars k and m, $k(mu) = (km)u$
10. For scalar 1, $1u = u$

Example: Let $V = \{\mathbf{0}\}$. Define that $\mathbf{0}+\mathbf{0}=\mathbf{0}$ and $k\cdot\mathbf{0}=\mathbf{0}$ for all scalar k

$\Rightarrow$ the zero vector space

Example Let $V=\mathbb{R}^n$. Define:
$$u+v=(u_1,u_2,\cdots,u_n)+(v_1,v_2,\cdots,v_n)=(u_1+v_1,u_2+v_2,\cdots,u_n+v_n)$$
$$ku=(ku_1,ku_2,\cdots,ku_n)$$
Then $V=\mathbb{R}^n$ is a vector space

Example: $V=\mathbb{R}^\infty=\{u=(u_1,u_2,\cdots,u_n,\cdots)\mid u_i\in\mathbb{R}, \text{ for all } i\}$

and $u+v=(u_1+v_1,u_2+v_2,\cdots,u_n+v_n,\cdots)$

$ku=(ku_1,ku_2,\cdots,ku_n,\cdots)$ for all $u,v\in\mathbb{R}^\infty$, all scalar k

Example Let V be the set of 2×2 matrices $V=\mathbb{R}^{2\times 2}$ such that
$$u+v=\begin{bmatrix}u_{11}&u_{12}\\u_{21}&u_{22}\end{bmatrix}+\begin{bmatrix}v_{11}&v_{12}\\v_{21}&v_{22}\end{bmatrix}=\begin{bmatrix}u_{11}+v_{11}&u_{12}+v_{12}\\u_{21}+v_{21}&u_{22}+v_{22}\end{bmatrix}$$
$$ku=k\begin{bmatrix}u_{11}&u_{12}\\u_{21}&u_{22}\end{bmatrix}=\begin{bmatrix}ku_{11}&ku_{12}\\ku_{21}&ku_{22}\end{bmatrix}$$
It's easy to see Axioms 1 & 6 hold.

For Axiom 2:
$$u+v=\begin{bmatrix}u_{11}&u_{12}\\u_{21}&u_{22}\end{bmatrix}+\begin{bmatrix}v_{11}&v_{12}\\v_{21}&v_{22}\end{bmatrix}=\begin{bmatrix}v_{11}&v_{12}\\v_{21}&v_{22}\end{bmatrix}+\begin{bmatrix}u_{11}&u_{12}\\u_{21}&u_{22}\end{bmatrix}$$
$$=v+u$$
For Axiom 4:

Let $\mathbf{0}=\begin{bmatrix}0&0\\0&0\end{bmatrix}$

then $\mathbf{0}+u=\begin{bmatrix}0&0\\0&0\end{bmatrix}+\begin{bmatrix}u_{11}&u_{12}\\u_{21}&u_{22}\end{bmatrix}=u$

For Axiom 5:

For $u\in V$, let $-u:=\begin{bmatrix}-u_{11}&-u_{12}\\-u_{21}&-u_{22}\end{bmatrix}$

then $u+(-u)=\begin{bmatrix}0&0\\0&0\end{bmatrix}=\mathbf{0}$

also, $(-u)+u=\mathbf{0}$

For Axiom 10:
$$1\cdot u=1\cdot\begin{bmatrix}u_{11}&u_{12}\\u_{21}&u_{22}\end{bmatrix}=\begin{bmatrix}u_{11}&u_{12}\\u_{21}&u_{22}\end{bmatrix}=u$$

Example (Vector space of real-valued functions)

Let V be the set of real valued functions, $F(-\infty, \infty)$

$f = f(x)$, $x \in (-\infty, \infty)$

$g = g(x)$ $\qquad f, g : (-\infty, \infty) \mapsto \mathbb{R}$

$f, g \in V$

then for any scalar k,

$$\left.\begin{aligned}(f+g)(x) &:= f(x) + g(x)\\ (k \cdot f)(x) &= k \cdot f(x)\end{aligned}\right\} \Rightarrow \text{Axioms 1 \& 6 hold}$$

For Axiom 4:

let $g = g(x) = 0$

then for each $f \in F(-\infty, \infty)$, $(f+g)(x) = f(x) + 0 = f(x)$

For Axiom 5:

for each $f \in F(-\infty, \infty)$,

define $-f := -f(x)$

For Axiom 2,

$$(f+g)(x) = \underbrace{f(x)}_{\text{reals}} + \underbrace{g(x)}_{\text{reals}} = g(x) + f(x) = (g+f)(x)$$

(counter-example)

Example: Let $V = \mathbb{R}^2$ and define that $u + v = (u_1 + v_1, u_2 + v_2)$

and let $ku = (ku_1, 0)$

Then V is NOT a vector space with the above operations.

$(\because 1u = 1 \cdot (u_1, u_2) = (u_1, 0) \neq u)$

4.1.1

Theorem Let V be a vector space and $u \in V$, k is a scalar.

Then : (a) $0u = \mathbf{0}$ $\qquad 0u + 0u = (0+0)u = 0u$

$0u$ has a negative $-0u$

$\Rightarrow [0u + 0u] + (-0u) = 0u + (-0u)$

$\Rightarrow 0u + (0u + (-0u)) = \mathbf{0}$

$\Rightarrow 0u + \mathbf{0} = \mathbf{0}$

$\Rightarrow 0u = \mathbf{0}$

(b) $k\mathbf{0} = \mathbf{0}$

(c) $(-1)u = -u$

$u + (-1)u = 1u + (-1)u = (1 + (-1))u = 0u = \mathbf{0}$

(d) If $ku = \mathbf{0}$, then $k = 0$ or $u = \mathbf{0}$

§ Subspaces

$W \subseteq V$, V is a vector space and

W is also a vector space under the addition and scalar multiplication defined on V

Then we say that W is a subspace of V

4.2.1

Theorem If $W \neq \phi$ is a nonempty set of vectors in a vector space V, then W is a subspace of V if and only if

(a) for each $u, v \in W$, $u+v \in W$

and (b) for any scalar k and $u \in W$, $ku \in W$

(proof)

$(\Rightarrow)$ trivial.

$(\Leftarrow)$: We have already Axioms 1 and 6

Axioms 2, 3, 7, 8, 9, 10 are inherited from V

We only need to show Axioms 4 and 5 hold in W

Let $u \in W$

by (b) we have $ku \in W$ for any scalar k

$\Rightarrow 0u = 0$ and $(-1)u = -u \in W$

$\therefore$ Axioms 4 and 5 hold in W ※

Example Let V be a vector space and $W = \{0\}$ is a subset of V

$\Rightarrow$ W is closed under addition and scalar multiplication

$0 + 0 = 0 \in W$

$k0 = 0 \in W$

W is the zero subspace of V

Example: Lines through the origin are subspaces of $\mathbb{R}^2$ and $\mathbb{R}^3$.

Example: Planes through the origin are subspaces of $\mathbb{R}^3$.

Example: Let $W = \{(x,y) \mid x,y \in \mathbb{R},\ x \geq 0 \text{ and } y \geq 0\}$

Then W is NOT a subspace of $\mathbb{R}^2$

$v = (1,1) \in W$, but $(-1)v = (-1,-1) \notin W$

Example: The set of real symmetric $n \times n$ matrices is a subspace of $\mathbb{R}^{n \times n} \to M_{nn}$ in the textbook

Example: The set of
- upper triangular matrices
- lower triangular matrices
- diagonal matrices

are all subspaces of $\mathbb{R}^{n\times n} \to M_{nn}$

Example: The set of $\boxed{\text{continuous}}$ functions on $(-\infty, \infty)$ is a subspace of $F(-\infty, \infty)$

$\because$ sum of continuous functions is still continuous
a constant times a continuous function is still continuous.

Example $W = \{p(x) = a_0 + a_1 x + \cdots + a_n x^n \mid a_0, a_1, \cdots, a_n \text{ are constants}\}$

$\Rightarrow$ closed under addition and scalar multiplication

$\therefore W$ is a subspace of $F(-\infty, \infty)$

$\overset{\shortparallel}{P_\infty}$

Example (Subspace Test)

Determine whether the following set of matrices is a subspace of M_{22}

① $U = \left\{ \begin{bmatrix} x & 0 \\ 2x & y \end{bmatrix} \middle| \ x, y \in \mathbb{R} \right\}$

② $W = \left\{ A \in M_{22} \middle| \ A \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$

(sol):

① Let $A, B \in U$, $A = \begin{bmatrix} a & 0 \\ 2a & b \end{bmatrix}$, $B = \begin{bmatrix} c & 0 \\ 2c & d \end{bmatrix}$

for some $a, b, c, d \in \mathbb{R}$

Consider $A+B = \begin{bmatrix} a+c & 0 \\ 2(a+c) & b+d \end{bmatrix} \in U$

also, for $k \in \mathbb{R}$, $kA = \begin{bmatrix} ka & 0 \\ 2ka & kb \end{bmatrix} \in U$

$\therefore$ U is a subspace of M_{22}

② Consider $A = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$

$\because$ $A \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\therefore A \in W$

However, $(2A) \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix} \neq \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\therefore$ $2A \notin W$

Thus, W is NOT a subspace of M_{22} ※

Example (Subspace Test)

Determine whether U and W are subspaces of $\mathbb{P}_2$:

① $U = \{ p = 1 + ax - ax^2 \mid a \in \mathbb{R} \}$

② $W = \{ p \in \mathbb{P}_2 \mid p(2) = 0 \}$

(sol): ① Consider $p = 1 + x - x^2$, $q = 1 + 2x - 2x^2$

then $p + q = 2 + 3x - 3x^2 \notin U$

$\therefore U$ is NOT closed under addition

$\Rightarrow U$ is NOT a subspace of $\mathbb{P}_2$

② Let $p, q \in W$ be two polynomials in W

Let k be any scalar, then

$(p+q)(2) = p(2) + q(2) = 0 \quad \therefore p+q \in W$

$(kp)(2) = k \cdot p(2) = k \cdot 0 = 0 \quad \therefore kp \in W$

$\therefore W$ is a subspace of $\mathbb{P}_2$. ✫

4.2.2 ✫✫

Theorem If $W_1, W_2, \cdots, W_r$ are all subspaces of a vector space V, then $W_1 \cap W_2 \cap \cdots \cap W_r$ is also a subspace of V

(proof): Let $W = W_1 \cap W_2 \cap \cdots \cap W_r \neq \phi$ ($\because 0 \in W$)

Let $u, v \in W$ be two vectors in W

$\therefore u \in W_i$, $v \in W_i$ for each $i = 1, 2, \cdots, r$

Since W_i is a subspace of V for each i.

$\therefore u + v$ and ku are in W_i for each i, any scalar k

$\Rightarrow u + v \in W$ and $ku \in W$

$\therefore W$ is a subspace of V ✫

4.2.3

Theorem The solution set of $A\mathbf{x} = \mathbf{0}$ of m equations in n unknowns is a subspace of $\mathbb{R}^n$.

(proof):

Let W be the solution set of $A\mathbf{x} = \mathbf{0}$

$W \neq \phi$ ($\because$ there is at least the trivial solution)

Let $\mathbf{x}_1, \mathbf{x}_2 \in W$

Since $\mathbf{x}_1, \mathbf{x}_2$ are solutions of $A\mathbf{x} = \mathbf{0}$

$\therefore A\mathbf{x}_1 = \mathbf{0}$ and $A\mathbf{x}_2 = \mathbf{0}$

$A(\mathbf{x}_1 + \mathbf{x}_2) = A\mathbf{x}_1 + A\mathbf{x}_2 = \mathbf{0} + \mathbf{0} = \mathbf{0}$

$\therefore W$ is closed under addition.

Next, $A(k\mathbf{x}_1) = kA\mathbf{x}_1 = k\mathbf{0} = \mathbf{0}$, for any scalar k

$\therefore W$ is closed under scalar multiplication

Thus, W is a subspace of $\mathbb{R}^n$. ∎

Note

Let $T_A : \mathbb{R}^n \mapsto \mathbb{R}^m$, $T_A(\mathbf{x}) = A\mathbf{x}$

The solution space of $A\mathbf{x} = \mathbf{0}$

are vectors mapped into the zero vector in $\mathbb{R}^m$

We call this set of vectors the kernel of T_A

Theorem

If $A \in \mathbb{R}^{m \times n}$, then the kernel of $T_A : \mathbb{R}^n \mapsto \mathbb{R}^m$, $T_A(\mathbf{x}) = A\mathbf{x}$, is a subspace of $\mathbb{R}^n$.

§ Spanning Sets

Defininition [linear Combination]

Given vectors $v_1, v_2, \cdots, v_r \in V$ for a vector space V

if $w = k_1 v_1 + k_2 v_2 + \cdots + k_r v_r$, for scalars $k_1, k_2, \cdots, k_r$

then w is said to be a linear combination of $v_1, v_2, \cdots, v_r$

4.3.1.

Theorem If $S = \{w_1, w_2, \cdots, w_r\} \subseteq V$, $S \neq \phi$ and V is a vector space. Then :

(a) The set W of all linear combinations of vectors in S is a subspace of V

(b) W is the smallest subspace of V that contains all the vectors in S

(proof)

(a) Let $W = \{ k_1 w_1 + k_2 w_2 + \cdots + k_r w_r \mid k_1, k_2, \cdots, k_r \text{ are scalars} \}$

Let $u, v \in W$, $u = c_1 w_1 + c_2 w_2 + \cdots + c_r w_r$

$v = d_1 w_1 + d_2 w_2 + \cdots + d_r w_r$

$\therefore u + v = (c_1 + d_1) w_1 + (c_2 + d_2) w_2 + \cdots + (c_r + d_r) w_r$

is a linear combination of vectors in $S \Rightarrow u + v \in W$

consider $\alpha \in \mathbb{R}$ is a scalar

$\alpha u = c_1 \alpha w_1 + c_2 \alpha w_2 + \cdots + c_r \alpha w_r \in W$

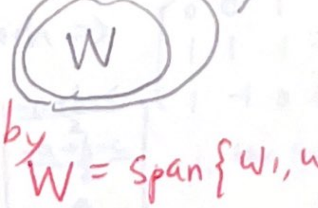
Thus, W is a subspace of V

(b) Let W' be any subspace of V that contains all vectors in S

$\because W'$ is closed under addition and scalar multiplication

$\therefore W \supseteq S$ and for each $v \in W$, say $v = k_1 w_1 + k_2 w_2 + \cdots + k_r w_r$

we also have $v \in W' \Rightarrow W \subseteq W'$



We denote by $W = \text{span}\{w_1, w_2, \cdots, w_r\}$ or $W = \text{span}(S)$

Example consider the stand unit vectors in $\mathbb{R}^n$:

$e_1 = (1,0,0,\cdots,0)$, $e_2 = (0,1,0,\cdots,0)$, $\cdots$, $e_n = (0,0,\cdots,1)$

any $v = (v_1, v_2, \cdots, v_n) \in \mathbb{R}^n$ is a linear combination of $e_1, e_2, \cdots, e_n$

$$v = v_1 e_1 + v_2 e_2 + v_3 e_3 + \cdots + v_n e_n$$

Example : Spanning Set for P_n

The polynomials : 1,

x,

x_2,

$\vdots$

x^n

consider any polynomial in P_n : $p = a_0 + a_1 x + \cdots + a_n x^n$

$\therefore P_n = span\{1, x, x^2, \cdots, x^n\}$

Example Suppose we have $u = (1,2,-1)$, $v = (6,4,2) \in \mathbb{R}^3$

Show that $w = (9,2,7)$ is a linear combination of u and v, and $w' = (4,-1,8)$ is NOT.

(sol): Solve $(9,2,7) = k_1(1,2,-1) + k_2(6,4,2)$

$\Rightarrow k_1 = -3,\ k_2 = 2$

$\therefore (9,2,7) \in span(\{u, v\})$

But we cannot find any k_1, k_2 for which

$w' = k_1 u + k_2 v$

$\Rightarrow w' = (4,-1,8) \notin span(\{u, v\})$

Example (Test for spanning)

Let $v_1 = (1,1,2)$, $v_2 = (1,0,1)$, $v_3 = (2,1,3)$

Do v_1, v_2, and v_3 span the vector space $\mathbb{R}^3$?

(sol): Consider any $b = (b_1, b_2, b_3) \in \mathbb{R}^3$

Suppose that $b = k_1 v_1 + k_2 v_2 + k_3 v_3$ for scalars k_1, k_2, k_3

$\Rightarrow (b_1, b_2, b_3) = k_1(1,1,2) + k_2(1,0,1) + k_3(2,1,3)$

$$\Rightarrow \begin{aligned} k_1 + k_2 + 2k_3 &= b_1 \\ k_1 \quad\quad + k_3 &= b_2 \\ 2k_1 + k_2 + 3k_3 &= b_3 \end{aligned} \quad \Leftrightarrow \quad \underbrace{\begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix}}_{A} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

But we can compute $\det(A) = 0$

Example Determine whether S spans $\mathbb{P}_2$:

(a) $S = \{1 + x + x^2, -1 - x, 2 + 2x + x^2\}$

(b) $S = \{x + x^2, x - x^2, 1 + x, 1 - x\}$

(Sol):

(a) $k_1(1 + x + x^2) + k_2(-1 - x) + k_3(2 + 2x + x^2) = a + bx + cx^2$

for any vector $p = a + bx + cx^2 \in \mathbb{P}_2$

$\Rightarrow (k_1 - k_2 + 2k_3) + (k_1 - k_2 + 2k_3)x + (k_1 + k_3)x^2 = a + bx + cx^2$

$$\Rightarrow \underbrace{\begin{bmatrix} 1 & -1 & 2 \\ 1 & -1 & 2 \\ 1 & 0 & 1 \end{bmatrix}}_{A} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

But $\det(A) = 0$ $\therefore$ For some choices of $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$, the system is inconsistent

$\Rightarrow S$ does NOT span $\mathbb{P}_2$

(b) Similar approach $\Rightarrow \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

$$\Rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & \frac{-a+b+c}{2} \\ 0 & 1 & 0 & 0 & \frac{-a+b-c}{2} \\ 0 & 0 & 1 & -1 & a \end{array}\right]$$

$\therefore$ the system is consistent for every choice of a, b, c

$\Rightarrow \text{span}(S) = \mathbb{P}_2$ *

Note : The spanning sets are NOT unique.

e.g., $(0,1)$ and $(1,0)$ for $\mathbb{R}^2$

$(1,2)$ and $(3,4)$ for $\mathbb{R}^2$,

⋮

(4.3.2)

Theorem If $S_1 = \{v_1, v_2, \cdots, v_r\}$ and $S_2 = \{w_1, w_2, \cdots, w_k\}$ are nonempty set of vectors in a vector space V then $\text{span}(S_1) = \text{span}(S_2)$ if and only if v is a linear combination of vectors in S_2 for each $v \in S_1$
