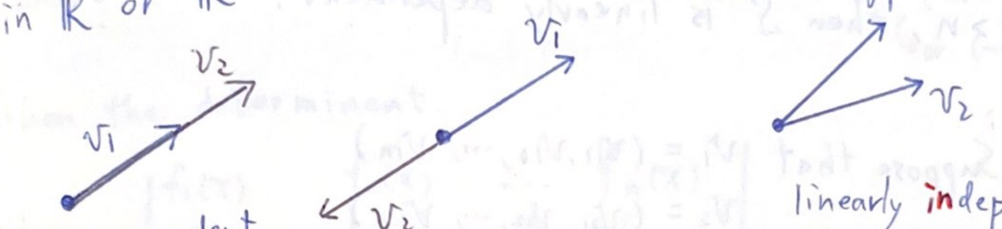
Remark on § linear independency

in $\mathbb{R}^2$ or $\mathbb{R}^3$



linearly dependent   linearly dependent   linearly independent

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Definition

$S = \{v_1, v_2, \cdots, v_r\}$ is linearly independent :

if no vector in S can be expressed as a linear combination of the others.

Theorem A nonempty set $S = \{v_1, v_2, \cdots, v_r\}$ in a vector space V is linearly independent if and only if the coefficients satisfying

$$k_1 v_1 + k_2 v_2 + \cdots + k_r v_r = \mathbf{0}$$

are $k_1 = k_2 = \cdots = k_r = 0$

Theorem 4.4.2

(a) A set with finitely many vectors that contains $\mathbf{0}$ is linearly dependent.

(b) A set with exactly two vectors is linearly independent if and only if neither vector is a scalar multiple of the other.

4.4.3

Theorem Let $S = \{v_1, v_2, \cdots, v_r\}$ be a set of vectors in $\mathbb{R}^n$. If $r > n$, then S is linearly dependent.

(proof):

Suppose that
$$v_1 = (v_{11}, v_{12}, \cdots, v_{1n})$$
$$v_2 = (v_{21}, v_{22}, \cdots, v_{2n})$$
$$\vdots$$
$$v_r = (v_{r1}, v_{r2}, \cdots, v_{rn})$$

Consider $k_1 v_1 + k_2 v_2 + \cdots + k_r v_r = 0$

$$\Rightarrow \begin{cases} v_{11} k_1 + v_{21} k_2 + \cdots + v_{r1} k_r = 0 \\ v_{12} k_1 + v_{22} k_2 + \cdots + v_{r2} k_r = 0 \\ \vdots \\ v_{1n} k_1 + v_{2n} k_2 + \cdots + v_{rn} k_r = 0 \end{cases}$$

it's a homogeneous system of n equations and r unknowns, $r > n$

$\Rightarrow$ the system has nontrivial solutions

$\therefore S$ is linearly dependent #

We also have linear dependence of "functions".

Example: $f_1 = \sin^2 x$, $f_2 = \cos^2 x$, $f_3 = 5$

$$\because 5f_1 + 5f_2 - f_3 = 5\sin^2 x + 5\cos^2 x - 5$$
$$= 5(\sin^2 x + \cos^2 x) - 5 = 0$$

$\therefore \{f_1, f_2, f_3\}$ is linearly dependent.

(Wronskian) If $f_1 = f_1(x)$, $f_2 = f_2(x)$, ..., $f_n = f_n(x)$ are functions that are $n-1$ times differentiable on $[a, b]$,
↳ or $(-\infty, \infty)$

~~where $f_1, f_2, \cdots, f_n : [a,b] \longmapsto$~~

then the determinant

$$W(x) = \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & & f_n^{(n-1)}(x) \end{vmatrix}$$

is called the Wronskian of $f_1, f_2, \cdots, f_n$.

Let's go deeper!

Suppose that $f_1, f_2, \cdots, f_n$ are linearly dependent.

Then $k_1 f_1 + k_2 f_2 + \cdots + k_n f_n = 0$ is satisfied by not-all-zero coefficients $k_1, k_2, \cdots, k_n$.

$k_1 f_1(x) + k_2 f_2(x) + \cdots + k_n f_n(x) = 0$ is satisfied for all x

↳ taking differentiations

$$k_1 f_1(x) + k_2 f_2(x) + \cdots + k_n f_n(x) = 0$$

$$k_1 f_1'(x) + k_2 f_2'(x) + \cdots + k_n f_n'(x) = 0$$

$$\vdots$$

$$k_1 f^{(n-1)}(x) + k_2 f^{(n-1)}(x) + \cdots + k_n f_n^{(n-1)}(x) = 0$$

$$\Rightarrow \begin{bmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & & f_n^{(n-1)}(x) \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$ has a nontrivial solution for all x

$\Rightarrow W(x) = 0$ for all x

n 個未知數

n-1 個方程式

<u>Example</u>. Use the Wronskian to show that $f_1 = x$, $f_2 = \sin x$ are linearly independent vectors in $C^{\infty}(-\infty, \infty)$

↳ infinitely differentiable on $(-\infty, \infty)$

(sol):

The Wronskian is

$$W(x) = \begin{vmatrix} x & \sin x \\ 1 & \cos x \end{vmatrix} = x\cos x - \sin x$$

$$\because W\left(\tfrac{\pi}{2}\right) = \tfrac{\pi}{2}\cos\left(\tfrac{\pi}{2}\right) - \sin\left(\tfrac{\pi}{2}\right) = -1 \neq 0$$

$\therefore f_1, f_2$ are linearly independent ※

<u>Example</u> Prove that $\{\sin x, \cos x, x\sin x\}$ is linearly independent.

(暨大資工91)

(proof):

The Wronskian is

$$W(x) = \begin{vmatrix} \sin x & \cos x & x\sin x \\ \cos x & -\sin x & \sin x + x\cos x \\ -\sin x & -\cos x & 2\cos x - x\sin x \end{vmatrix}$$

Take $x = 0$,

$$W(0) = \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 2 \end{vmatrix} = -2 \neq 0$$

$\therefore \{\sin x, \cos x, x\sin x\}$ is linearly independent ※

Remark on spanning set

Given $u_1=(1,2,1)$, $u_2=(-2,-4,-2)$, $u_3=(0,2,3)$, $u_4=(2,0,-3)$, $u_5=(-3,8,16)$, determine whether $u=(2,6,8)$ can be expressed as a linear combination of u_1,u_2,u_3,u_4,u_5.

(Sol): Let
$$A=\left[\begin{array}{ccc} 1 & 2 & 1 \\ -2 & -4 & -2 \\ 0 & 2 & 3 \\ 2 & 0 & -3 \\ -3 & 8 & 16 \\ \hline 2 & 6 & 8 \end{array}\right] \xrightarrow[r_{15}^{(3)},\ r_{16}^{(2)}]{r_{12}^{(2)},\ r_{14}^{(-2)}} B=\left[\begin{array}{ccc} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 2 & 3 \\ 0 & -4 & -5 \\ 0 & 14 & 19 \\ \hline 0 & 2 & 6 \end{array}\right]$$

$$\xrightarrow{r_{34}^{(2)},\ r_{35}^{(-7)},\ r_{36}^{(-1)}} C=\left[\begin{array}{ccc} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & -2 \\ \hline 0 & 0 & 3 \end{array}\right] \xrightarrow{r_{45}^{(2)},\ r_{46}^{(-3)}} D=\left[\begin{array}{ccc} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right]$$

So we have

$$-3C_4+C_6=0$$

$$C_4=2B_3+B_4,\quad C_6=-B_3+B_6$$

$$\therefore -3(2B_3+B_4)+(-B_3+B_6)=0$$

$$\Rightarrow -7B_3-3B_4+B_6=0$$

And, $B_3=A_3$, $B_4=-2A_1+A_4$, $B_6=-2A_1+A_6$

$$\therefore -7A_3-3(-2A_1+A_4)+(-2A_1+A_6)=0$$

$$\Rightarrow 4A_1+0A_2-7A_3-3A_4+0\cdot A_5+A_6=0$$

$$\Rightarrow 4u_1+0u_2-7u_3-3u_4+0\cdot u_5+u=0$$

$$\therefore u=-4u_1+0u_2+7u_3+3u_4+0\cdot u_5 \quad \#$$

Note: Never do row interchange!

Example: Determine whether the matrix $\begin{bmatrix} -1 & 7 \\ 8 & -1 \end{bmatrix}$ is a linear combination of $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$, $\begin{bmatrix} 2 & -3 \\ 0 & 2 \end{bmatrix}$, and $\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$

(98雲科大資工)

sol):

$$\left[\begin{array}{cccc} 1 & 0 & 2 & 1 \\ 2 & -3 & 0 & 2 \\ 0 & 1 & 2 & 0 \\ \hline -1 & 7 & 8 & -1 \end{array}\right] \xrightarrow{r_{12}^{(-2)},\ r_{14}^{(1)}} \left[\begin{array}{cccc} 1 & 0 & 2 & 1 \\ 0 & -3 & -4 & 0 \\ 0 & 1 & 2 & 0 \\ \hline 0 & 7 & 10 & 0 \end{array}\right]$$

$$\xrightarrow{r_{23}^{(\frac{1}{3})},\ r_{24}^{(\frac{7}{3})}} \left[\begin{array}{cccc} 1 & 0 & 2 & 1 \\ 0 & -3 & -4 & 0 \\ 0 & 0 & \frac{2}{3} & 0 \\ \hline 0 & 0 & \frac{2}{3} & 0 \end{array}\right] \xrightarrow{r_{34}^{(-1)}} \left[\begin{array}{cccc} 1 & 0 & 2 & 1 \\ 0 & -3 & -4 & 0 \\ 0 & 0 & \frac{2}{3} & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right]$$

$\therefore$ The answer is "yes"

Exercise (3-34) Let $u=(3,0,2)$, $v=(0,1,1)$, and $w=(-3,1,0)$. Is $(0,3,5)$ in span$(\{u,v,w\})$? (98中正資工)

Exercise (3-35) Let $v_1=(1,-1,-2)$, $v_2=(5,-4,-7)$, $v_3=(-3,1,0)$, and $y=(-4,3,h)$. Find the value of h such that y is in span$(\{v_1,v_2,v_3\})$. (91輔大資工)

How to determine if a set of vectors spans the vector space?

Example: Determine whether the vectors $v_1 = (1,1,2)$, $v_2 = (1,0,1)$, and $v_3 = (2,1,3)$ spans $\mathbb{R}^3$

sol): Let $b = (b_1, b_2, b_3)$ be an arbitrary vector in $\mathbb{R}^3$

Suppose that $b = k_1 v_1 + k_2 v_2 + k_3 v_3$

$\therefore (b_1, b_2, b_3) = (k_1 + k_2 + 2k_3,\ k_1 + k_3,\ 2k_1 + k_2 + 3k_3)$

$$\Rightarrow \begin{aligned} k_1 + k_2 + 2k_3 &= b_1 \\ k_1 + k_3 &= b_2 \\ 2k_1 + k_2 + 3k_3 &= b_3 \end{aligned}, \quad A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix}$$

The system is consistent if and only if $\det(A) \neq 0$

But we have $\det(A) = 0 \longrightarrow$ check by yourself.

$\therefore v_1, v_2, v_3$ do not span $\mathbb{R}^3$

Example: Determine whether $S = \{x + x^2, x - x^2, 1 + x, 1 - x\}$ spans P_2.

sol): An arbitrary vector in P_2 is of the form $a + bx + cx^2$

$\therefore$ Let $k_1(x + x^2) + k_2(x - x^2) + k_3(1 + x) + k_4(1 - x) = a + bx + cx^2$

$$\Rightarrow \begin{aligned} k_3 + k_4 &= a \\ k_1 + k_2 + k_3 - k_4 &= b \\ k_1 - k_2 &= c \end{aligned}, \quad A = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & -1 & 0 & 0 \end{bmatrix} \to \text{not square!}$$

Consider the augmented matrix $\begin{bmatrix} 0 & 0 & 1 & 1 & a \\ 1 & 1 & 1 & -1 & b \\ 1 & -1 & 0 & 0 & c \end{bmatrix}$

$\to$ reduced row echelon form:

$$\begin{array}{c} \begin{bmatrix} 1 & 0 & 0 & 0 & \frac{-a+b+c}{2} \\ 0 & 1 & 0 & 0 & \frac{-a+b-c}{2} \\ 0 & 0 & 1 & -1 & a \end{bmatrix} \\ \begin{array}{cccc} k_1 & k_2 & k_3 & k_4 \end{array} \end{array} \to \text{check by yourself.}$$

$\therefore$ the system is consistent for every choice a, b, and $c \Rightarrow \operatorname{span}(S) = P_2$ #