§ Basis for a Vector Space

Definition: If $S = \{v_1, v_2, \cdots, v_n\}$ is a set of vectors in a finite-dimensional vector space V, then we say S is a basis for V if:

(a) S spans V

& (b) S is linearly independent

Example standard unit vectors $e_1, e_2, \cdots, e_n$ for R^n

$\Rightarrow$ standard basis for R^n

Example: $S = \{1, x, x^2, \cdots, x^n\}$ is a basis for P_n

$\Rightarrow$ standard basis for P_n

Example: $M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, M_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, M_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, M_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

form a basis for $\underbrace{M_{22}}_{2\times 2 \text{ matrices}}$

Theorem [uniqueness of basis representation]

If $S = \{v_1, v_2, \cdots, v_n\}$ is a basis for a vector space V, then every $v \in V$ can be expressed as $v = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$ in "exactly one way".

(proof): S is a basis for $V \Rightarrow v$ can be expressed as a linear combination of vectors in S

Suppose that

$$v = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n \quad \cdots ①$$

$$\text{and } v = d_1 v_1 + d_2 v_2 + \cdots + d_n v_n \quad \cdots ②$$

① − ② $\Rightarrow (c_1 - d_1) v_1 + (c_2 - d_2) v_2 + \cdots + (c_n - d_n) v_n = 0$

$\because S$ is a basis $\Rightarrow S$ is a linearly independent set

$\therefore c_1 - d_1 = 0, \; c_2 - d_2 = 0, \cdots, c_n - d_n = 0$

That is, $c_1 = d_1, \; c_2 = d_2, \cdots, c_n = d_n$, So the representation is unique #

Since the basis representation is unique:

Definition Let $S = \{v_1, v_2, \cdots, v_n\}$ be an ordered basis for a vector space V, and $v = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$

Then the scalars $c_1, c_2, \cdots, c_n$ are the coordinates of v relative to the basis S

The coordinate "vector" of v relative to S:

$$[v]_S = (c_1, c_2, \cdots, c_n)$$

$$\text{or } [v]_S = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

Example: $P(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n$

standard basis for P_n : $\{1, x, x^2, \cdots, x^n\} := S$

then $[P]_S = (c_0, c_1, \cdots, c_n)$

Example: $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the standard basis for M_{22}:

$$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

then $[B]_S = (a, b, c, d)$

Example: Suppose we have $\{v_1 = (1, 2, 1),$
$v_2 = (2, 9, 0),$
$v_3 = (3, 3, 4)\} = S$

S is a basis for $\mathbb{R}^3$. Consider $[v]_S = (-1, 3, 2)$

Then $v = (-1) \cdot (1, 2, 1) + 3 \cdot (2, 9, 0) + 2 \cdot (3, 3, 4)$
$= (11, 31, 7)$ ✱

If we have $v = (5, -1, 9)$

we can assume that $v = c_1 v_1 + c_2 v_2 + c_3 v_3$

$\Rightarrow (5, -1, 9) = c_1 (1, 2, 1) + c_2 (2, 9, 0) + c_3 (3, 3, 4)$

Solving the system $\Rightarrow c_1 = 1, c_2 = -1, c_3 = 2$

Then we know that $[v]_S = (1, -1, 2)$ ✱

Dimension and Change of Basis

4.6.2
Theorem Let V be a finite-dimensional vector space and let $\{v_1, v_2, \cdots, v_n\}$ be any basis for V

(a) If a set in V has more than n vectors, then it is linearly dependent

(b) If a set in V has fewer than n vectors, then it does NOT span V

4.6.3
Theorem All bases for a finite-dimensional vector space have the same number of vectors.

What's the "dimension" of a vector space?

Definition: The dimension of a vector space V

$\Rightarrow \dim(V)$: the number of vectors in a basis for V

Note: $\dim(0) := 0$

↳ "is defined"

Example:

$\dim(\mathbb{R}^n) = n$

$\dim(P_n) = n+1$

$\dim(M_{mn}) = mn$

Example: If $S = \{v_1, v_2, \cdots, v_r\}$, then every vector in span(S) is expressible as linear combination of the vectors in S

If $v_1, v_2, \cdots, v_r$ are linearly independent

$\Rightarrow \{v_1, v_2, \cdots, v_r\}$ is a basis for span(S)

$\Rightarrow \dim(\text{span}(\{v_1, v_2, \cdots, v_r\})) = r$

Example (dimension of a solution space)

Find a basis for and the dimension of the solution space of

$$x_1 + 3x_2 - 2x_3 + 2x_5 = 0$$
$$2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 = 0$$
$$5x_3 + 10x_4 + 15x_6 = 0$$
$$2x_1 + 6x_2 + 8x_4 + 4x_5 + 18x_6 = 0$$

(sol): Solve the system above, we have:

$x_1 = -3r - 4s - 2t$
$x_2 = r$
$x_3 = -2s$
$x_4 = s$
$x_5 = t$
$x_6 = 0$

$\therefore (x_1, x_2, \cdots, x_6)$

$= (-3r-4s-2t, r, -2s, s, t, 0)$

$= r(-3, 1, 0, 0, 0, 0) + s(-4, 0, -2, 1, 0, 0) + t(-2, 0, 0, 0, 1, 0)$

$\therefore (-3, 1, 0, 0, 0, 0), (-4, 0, -2, 1, 0, 0), (-2, 0, 0, 0, 1, 0)$ spans the solution space

↳ Check that whether they are linearly independent

$\Rightarrow$ dimension $= 3$

Change of Basis (Recall)

Let V be a vector space of $\dim(V) = 2$

Let $\beta_1 = \{u_1, u_2\}$, $\beta_2 = \{w_1, w_2\}$ be two bases of V

Suppose that

$$\boxed{[u_1]_{\beta_2} = \begin{bmatrix} a \\ b \end{bmatrix}, \quad [u_2]_{\beta_2} = \begin{bmatrix} c \\ d \end{bmatrix}}$$

That is,

$$u_1 = a w_1 + b w_2$$

$$u_2 = c w_1 + d w_2$$

Now, let $v \in V$ be any vector, suppose that

$\boxed{[v]_{\beta_1} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}}$, that is, $v = k_1 u_1 + k_2 u_2$

$$\therefore v = k_1(a w_1 + b w_2) + k_2(c w_1 + d w_2)$$

$$= (k_1 a + k_2 c) w_1 + (k_1 b + k_2 d) w_2$$

Thus, $$[v]_{\beta_2} = \begin{bmatrix} k_1 a + k_2 c \\ k_1 b + k_2 d \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \boxed{\begin{bmatrix} k_1 \\ k_2 \end{bmatrix}}$$

$$= \underbrace{\begin{bmatrix} a & c \\ b & d \end{bmatrix}}_{[I]_{\beta_1}^{\beta_2}} [v]_{\beta_1}$$

Therefore,

$$[v]_{\beta_2} = [I]_{\beta_1}^{\beta_2} [v]_{\beta_1}$$

$$[v]_{\beta_1} = [I]_{\beta_2}^{\beta_1} [v]_{\beta_2}$$

Note: $[I]_{\beta_1}^{\beta_2} [I]_{\beta_2}^{\beta_1} = I$ ⇝ Later it will be used for an efficient algorithm.

Example Consider two bases $B = \{u_1, u_2\}$, $B' = \{u_1', u_2'\}$ for $\mathbb{R}^2$

where $u_1 = (1,0)$, $u_2 = (0,1)$, $u_1' = (1,1)$, $u_2' = (2,1)$

Find $[I]_B^{B'}$ and $[I]_{B'}^{B}$

(sol): $u_1 = -u_1' + u_2'$
$u_2 = 2u_1' - u_2'$ $\Rightarrow [I]_B^{B'} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$

$u_1' = u_1 + u_2$
$u_2' = 2u_1 + u_2$ $\Rightarrow [I]_{B'}^{B} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$

Example Given $[v]_B = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$, B and B' are two bases as above

Then, $[v]_{B'} = [I]_B^{B'} [v]_B = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} -3 \\ 5 \end{bmatrix}$

$= \begin{bmatrix} 13 \\ -8 \end{bmatrix}$

$\boxed{\begin{array}{c} \langle \text{An Efficient Method of Computing Transition Matrices} \rangle \\ [\text{new basis} \mid \text{old basis}] \xrightarrow[\text{operations}]{\text{row}} [I \mid [I]_{\text{old}}^{\text{new}}] \end{array}}$

Example: Let $B = \{u_1, u_2\}$, $B' = \{u_1', u_2'\}$ as above

$$[\text{new basis} \mid \text{old basis}] = \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ u_1' & u_2' & u_1 & u_2 \end{array}\right]$$

$\xrightarrow{\text{row operations}} [I \mid [I]_{\text{old}}^{\text{new}}] = \left[\begin{array}{cc|cc} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & -1 \end{array}\right]$

$\Rightarrow [I]_B^{B'} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$

4.6.3 (PLUS/MINUS)

Theorem Let $S \neq \phi$ be a set of vectors in a vector space V

(a) If S is linearly independent, and $v \in V$ is outside $\text{span}(S)$, then $S \cup \{v\}$ is still linearly independent.

(b) If $v \in S$ that is a linear combination of other vectors in S, then $S - \{v\}$ spans the same space as S

That is, $\text{span}(S) = \text{span}(S - \{v\})$

Example:

$P_1 = 1 - x^2$, $P_2 = 2 - x^2$, $P_3 = x^3$ are linearly independent?

(proof): ① Show that $S = \{P_1, P_2\}$ is linearly independent

② P_3 cannot be expressed as a linear combination of S

$\therefore S \cup \{P_3\} = \{P_1, P_2, P_3\}$ is a linearly independent set.

Recall

$w \in V$, $w = k_1 v_1 + k_2 v_2 + \cdots + k_r v_r$

where $k_1, k_2, \cdots, k_r$ are scalars,

w is a "linear combination" of $v_1, v_2, \cdots, v_r$

4.3.1

Theorem If $S = \{w_1, w_2, \cdots, w_r\}$ is a nonempty set of vectors in a vector space V, then

(a) W = all possible linear combinations of the vectors in S

W is a subspace of V

(b) The set W in (a) is the "smallest" subspace of V that contains all of the vectors in S

(sketch of the proof of Theorem 4.3.1):

(a) $u = c_1w_1 + c_2w_2 + \cdots + c_rw_r$, $v = k_1w_1 + k_2w_2 + \cdots + k_rw_r$

① $u + v = (c_1+k_1)w_1 + (c_2+k_2)w_2 + \cdots + (c_r+k_r)w_r \rightsquigarrow \in W$

② Let $t \in \mathbb{R}$,

$tu = (tc_1)w_1 + (tc_2)w_2 + \cdots + (tc_r)w_r \rightsquigarrow \in W$

(b) Let W' be any subspace of V that contains all of the vectors in S

$\because W'$ is a subspace of V

$\therefore W'$ is closed under addition and scalar multiplication

$\Rightarrow W'$ contains all linear combinations of the vectors in S

$\Rightarrow W' \supseteq W$

(proof of Theorem 4.6.2):

(a) Let $S' = \{w_1, w_2, \cdots, w_m\}$, $m > n$, be any set of m vectors in V

$\because S = \{v_1, v_2, \cdots, v_n\}$ is a basis of V

$\therefore$ Let each $w_i = a_{1i}v_1 + a_{2i}v_2 + \cdots + a_{ni}v_n$, for $1 \le i \le m$

Let $k_1w_1 + k_2w_2 + \cdots + k_mw_m = 0$ → Want to show that not all k_i's are zeros

$$[w_1|w_2|\cdots|w_m] = [v_1|v_2|\cdots|v_n]\begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}$$

$$\therefore [w_1|w_2|\cdots|w_m]\begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_m \end{bmatrix} = [v_1|v_2|\cdots|v_n]\begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_m \end{bmatrix} = 0$$

where $\begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_m \end{bmatrix}$ is a nontrivial solution of $\begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = 0$

未知數比方程式多

(b) Exercise for students!

4.6.5

Theorem Let S be a finite set of vectors in a finite-dimensional vector space V.

(a) If S spans V but is not a basis for V, then S can be reduced to a basis for V by removing appropriate vectors from S

$\rightarrow$ S is linear dependent $\rightarrow$ Some $v \in S$ is a linear combination of the others $\rightarrow$ remove v from S $\rightarrow$ ……

(b) If S is a linearly independent set that is not a basis for V, then S can be enlarged to a basis for V by inserting appropriate vectors into S.

$\rightarrow$ S fails to span V $\rightarrow$ some $v \in V$ is NOT in span(S) $\rightarrow$ insert v into S $\rightarrow$ ……

4.6.6

Theorem If W is a subspace of a finite-dimensional vector space V, then:

(a) W is finite-dimensional.

(b) $\dim(W) \leq \dim(V)$

(c) $W = V$ if and only if $\dim(W) = \dim(V)$

(sketch of the proof of Theorem 4.6.2(b)) "$m<n$"

$$w_1 = a_{11}v_1 + a_{21}v_2 + \cdots + a_{n1}v_n$$
$$w_2 = a_{12}v_1 + a_{22}v_2 + \cdots + a_{n2}v_n$$
$$\vdots$$
$$w_m = a_{1m}v_1 + a_{2m}v_2 + \cdots + a_{nm}v_n$$

where $\{v_1, v_2, \cdots, v_n\}$ is a basis of V

$$\Rightarrow [w_1 | w_2 | \cdots | w_m] = [v_1 | v_2 | \cdots | v_n] \underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & & a_{nm} \end{bmatrix}}_{A_{n\times m}}$$

Let $\bar{v} = b_1v_1 + b_2v_2 + \cdots + b_nv_n$ be any vector in V, $b_1, b_2, \cdots, b_n \in \mathbb{R}$

We want to show that $\exists\, k_1, k_2, \cdots, k_m \in \mathbb{R}$ such that

$$\underbrace{[w_1 | w_2 | \cdots | w_m]}\begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_m \end{bmatrix} = [v_1 | v_2 | \cdots | v_n]\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$\Rightarrow [v_1 | v_2 | \cdots | v_n] \overbrace{n\left\{\underbrace{\begin{bmatrix} \\ & A & \\ \\ \end{bmatrix}}_{m}\right.}\begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_m \end{bmatrix} = [v_1 | v_2 | \cdots | v_n]\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$\because$ unique representation of basis for any vector!

Solve $\begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_m \end{bmatrix}$ by the augmented matrix:

$$\left[\begin{array}{c|c} & b_1 \\ A & b_2 \\ & \vdots \\ & b_n \end{array}\right] \rightsquigarrow \left[\begin{array}{cccc|c} 1 & & 0 & & b_1' \\ & 1 & & & b_2' \\ 0 & & \ddots & 1 & \vdots \\ 0 & \cdots & \cdots & 0 & b_n' \end{array}\right] \Rightarrow \text{no solution for } k_1, k_2, \cdots, k_m$$

$\hookrightarrow$ let it $b_n' \neq 0$