Row Space, Column Space, and Null Space

$$A_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

→ row vector (r_1)

↓ column vector (c_2)

row space of A: (row(A))

the subspace of $\mathbb{R}^n$ spanned by the row vectors of A

column space of A: (col(A))

the subspace of $\mathbb{R}^m$ spanned by the column vectors of A

null space of A (null(A))

the solution space of $A\mathbf{x} = 0$

(a subspace of $\mathbb{R}^n$)

Important Concepts:

- For $A\mathbf{x} = b$, the relationships among the solutions, row space, column space, null space of A ?
- row space $\overset{?}{\longleftrightarrow}$ column space $\overset{?}{\longleftrightarrow}$ null space

Say $A = \begin{array}{c} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \\ \begin{matrix} c_1 & \quad c_2 & \qquad & c_n \end{matrix} \end{array}$, $\quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

Then $A\mathbf{x} = x_1 \cdot c_1 + x_2 \cdot c_2 + \cdots + x_n \cdot c_n = b$

$\therefore$ $A\mathbf{x} = b$ is consistent

$\Updownarrow$

b is expressible as a linear combination of the column vectors of A. $\Leftrightarrow b \in col(A)$

Example Let $A\mathbf{x} = b$, where $A = \begin{bmatrix} -1 & 3 & 2 \\ 1 & 2 & -3 \\ 2 & 1 & -2 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ -9 \\ -3 \end{bmatrix}$

Solve the system by Gaussian elimination:

$$\begin{cases} x_1 = 2 \\ x_2 = -1 \\ x_3 = 3 \end{cases}$$

$$\Rightarrow 2\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} + (-1)\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ -3 \end{bmatrix}$$

§ The relationship between $A\mathbf{x} = 0$ and $A\mathbf{x} = b$

Example:

$$\underbrace{\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 2 & 6 & 0 & 8 & 4 & 18 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow \text{homogeneous } ①$$

and

$$\underbrace{\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 2 & 6 & 0 & 8 & 4 & 18 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 5 \\ 6 \end{bmatrix} \rightarrow \text{non-homogeneous } ②$$

① $\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = r \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -4 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

② $\Rightarrow \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}}_{x} = \underbrace{r \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -4 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{x_h \text{: by a basis from null}(A)} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1/3 \end{bmatrix}}_{x_0}$

4.8.2

Theorem If x_0 is any solution of a consistent $A\mathbb{X} = b$, and if $S = \{v_1, v_2, \cdots, v_k\}$ is a basis for null(A), then every solution of $A\mathbb{X} = b$ can be expressed in the form

$$\mathbb{X} = x_0 + c_1 v_1 + c_2 v_2 + \cdots + c_k v_k .$$

Conversely, for all choices of scalars $c_1, c_2, \cdots, c_k$, the vector $\mathbb{X}$ in this formula is a solution of $A\mathbb{X} = b$.

x_0 : particular solution of $A\mathbb{X} = b$

x_h : general solution of $A\mathbb{X} = 0$

(sketch of the proof): Let $\mathbb{X}_0$ be any solution of $A\mathbb{X} = b$, $W = $ null(A)

($\Leftarrow$) Let $\mathbb{X} \in \mathbb{X}_0 + W$

$\Rightarrow \mathbb{X} = \mathbb{X}_0 + w$, where $A\mathbb{X}_0 = b$ and $Aw = 0$

$\Rightarrow A\mathbb{X} = A(\mathbb{X}_0 + w) = A\mathbb{X}_0 + Aw = b + 0 = b$

($\Rightarrow$) Let $\mathbb{X}$ be any solution of $A\mathbb{X} = b$. Let $w = \mathbb{X} - \mathbb{X}_0$

Note that $A(\mathbb{X} - \mathbb{X}_0) = A\mathbb{X} - A\mathbb{X}_0 = b - b = 0$

$\therefore w = \mathbb{X} - \mathbb{X}_0 \in$ null$(A) \Rightarrow \mathbb{X} \in \mathbb{X}_0 + W$

4.8.3

Theorem (a) Row equivalent matrices have the same row space.

(b) Row equivalent matrices have the same null space.

Note: A, B are row equivalent:

$$A \underset{\longleftarrow}{\longrightarrow} B$$

elementary row operations

sketch (a) elementary row operations $\begin{cases} ① \text{ scalar multiplication} \\ ② \text{ linear combinations} \end{cases}$

(b) elementary row operations DO NOT Change the solution space.

Note: Elementary row operations could change the column space of a matrix!!

Example: $A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \xrightarrow{R_{12}^{(-2)}} B = \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$

$\text{col}(A) = s\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $s \in \mathbb{R}$ $\qquad$ $\text{col}(B) = t\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $t \in \mathbb{R}$

☆☆ 4.8.4

Theorem If a matrix is in row echelon form, then the row vectors with the leading 1's (the nonzero row vectors) form a basis for the row space of the matrix, and the column vectors with the leading 1's of the row vectors form a basis for the column space of the matrix.

Remark: It finds the bases for row(A) and col(A) at the same time for a matrix A if A is in row echelon form!

Example Find bases for row(R) and col(R), where

$$R = \begin{bmatrix} 1 & -2 & 5 & 0 & 3 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

sol): Since R is already in row echelon form,

$r_1 = [1 \ -2 \ 5 \ 0 \ 3]$

$r_2 = [0 \ 1 \ 3 \ 0 \ 0]$

$r_3 = [0 \ 0 \ 0 \ 1 \ 0]$

$\{r_1, r_2, r_3\}$ is a basis for row(R)

And,

$$c_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad c_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad c_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$\{c_1, c_2, c_3\}$ is a basis for col(R). ☆

Exercise : Find a basis of row(A), where

$$A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{bmatrix}$$

4.8.5

Theorem If A and B are row equivalent matrices. Then:

(a) A given set of column vectors of A is linearly independent if and only if the corresponding column vectors of B are linearly independent.

(b) A given set of column vectors of A forms a basis for $col(A)$ if and only if the corresponding column vectors of B form a basis for $col(B)$

Example Find a basis for $row(A)$, where

$$A = \begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 2 & -5 & -3 & -2 & 6 \\ 0 & 5 & 15 & 10 & 0 \\ 2 & 6 & 18 & 8 & 6 \end{bmatrix}$$

(sol): Method 1: row echelon form of A

Method 2: find a basis for $col(A^T)$

$$A^T = \begin{bmatrix} 1 & 2 & 0 & 2 \\ -2 & -5 & 5 & 6 \\ 0 & -3 & 15 & 18 \\ 0 & -2 & 10 & 8 \\ 3 & 6 & 0 & 6 \end{bmatrix}$$

row operations $\rightsquigarrow$ $\begin{bmatrix} \boxed{1} & 2 & 0 & 2 \\ 0 & \boxed{1} & -5 & -10 \\ 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\therefore$ A basis for $col(A^T)$:

$$c_1 = \begin{bmatrix} 1 \\ -2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \quad c_2 = \begin{bmatrix} 2 \\ -5 \\ -3 \\ -2 \\ 6 \end{bmatrix}, \quad c_3 = \begin{bmatrix} 2 \\ 6 \\ 18 \\ 8 \\ 6 \end{bmatrix}$$

$\Rightarrow$ a basis for $row(A)$:

$r_1 = [1 \ -2 \ 0 \ 0 \ 3]$

$r_2 = [2 \ -5 \ -3 \ -2 \ 6]$

$r_3 = [2 \ 6 \ 18 \ 8 \ 6]$

Rank, Nullity, and the Fundamental Matrix Spaces

4.9.1
Theorem $\dim(\text{row}(A)) = \dim(\text{col}(A))$

Remark: It's true by previous examples. Right?

Let R be any row echelon form of A.

$\dim(\text{row}(A)) = \dim(\text{row}(R))$ 4.8.4

$\dim(\text{col}(A)) = \dim(\text{col}(R))$

Definition (Rank)

$$\text{rank}(A) := \dim(\text{row}(A)) = \dim(\text{col}(A))$$

Definition (Nullity)

$$\text{nullity}(A) := \dim(\text{null}(A))$$

Example: Find $\text{rank}(A)$ and $\text{nullity}(A)$, where

$$A = \begin{bmatrix} -1 & 2 & 0 & 4 & 5 & -3 \\ 3 & -7 & 2 & 0 & 1 & 4 \\ 2 & -5 & 2 & 4 & 6 & 1 \\ 4 & -9 & 2 & -4 & -4 & 7 \end{bmatrix}$$

(Sol): The reduced row echelon form of A:

$$\begin{bmatrix} 1 & 0 & -4 & -28 & -37 & 13 \\ 0 & 1 & -2 & -12 & -16 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore \dim(\text{row}(A)) = \dim(\text{col}(A)) = 2$

$\therefore \text{rank}(A) = 2$

Moreover,

$$x_1 - 4x_3 - 28x_4 - 37x_5 + 13x_6 = 0$$

$$x_2 - 2x_3 - 12x_4 - 16x_5 + 5x_6 = 0$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = r\begin{bmatrix} 4 \\ 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 28 \\ 12 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 37 \\ 16 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + u\begin{bmatrix} -13 \\ 5 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad r, s, t, u \in \mathbb{R}$$

$\therefore \text{nullity}(A) = 4$

Remark: For an $m \times n$ matrix A,

$$\operatorname{rank}(A) \le \min(m, n)$$

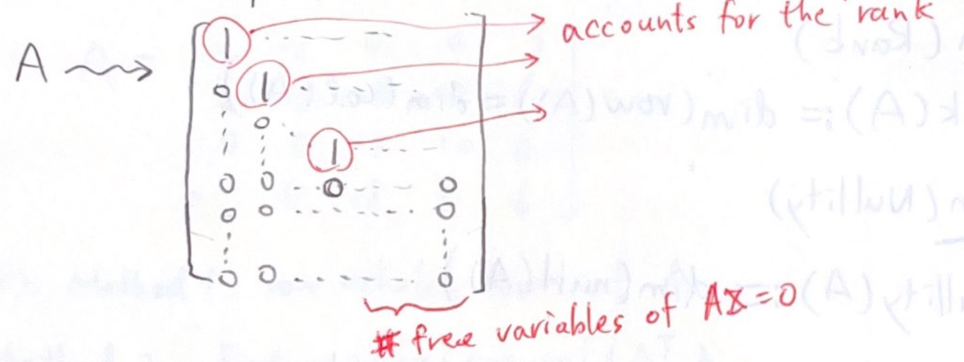
4.9.2

Theorem (The Dimension Theorem)

If A is a matrix with n columns, then

$$\operatorname{rank}(A) + \operatorname{nullity}(A) = n$$

(sketch of the proof):



4.9.3

Theorem If $A \in \mathbb{F}^{m \times n}$, then

(a) $\operatorname{rank}(A) =$ the number of leading variables in the general solution of $A\mathbf{x} = 0$

(b) $\operatorname{nullity}(A) =$ the number of parameters in the general solution of $A\mathbf{x} = 0$

Example: ① If $A \in \mathbb{F}^{5\times 7}$, $\operatorname{rank}(A) = 3$

then $\operatorname{nullity}(A) = 7 - \operatorname{rank}(A) = 4$

$\Rightarrow$ 4 parameters in the general solution of $A\mathbf{x} = 0$

② If $A \in \mathbb{F}^{5\times 7}$, $A\mathbf{x} = 0$ has a two-dimensional solution space

then $\operatorname{rank}(A) = 7 - \operatorname{nullity}(A) = 7 - 2 = 5$

Hence,

If $A\mathbf{x}=b$ is a consistent linear system of m equations in n unkowns

and if $\text{rank}(A)=r$,

then the general solution of $A\mathbf{x}=b$ contains $n-r$ parameters.

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§ Fundamental Spaces of a Matrix

$\text{row}(A)$, $\text{row}(A^T)$

$\text{col}(A)$, $\text{col}(A^T)$

$\text{null}(A)$, $\text{null}(A^T)$

Only the four of them are distinct!

$\Rightarrow$ fundamental spaces of A

4.9.5

Theorem For any matrix A, $\text{rank}(A)=\text{rank}(A^T)$

Therefore, for $A\in\mathbb{F}^{m\times n}$ (m rows, n columns)

$$\text{rank}(A^T)+\text{nullity}(A^T)=m$$

$$\Rightarrow \text{rank}(A)+\text{nullity}(A^T)=m$$

$\therefore$ If $\text{rank}(A)=r$, then:

$$\dim(\text{row}(A))=r,\ \dim(\text{col}(A))=r$$

$$\dim(\text{null}(A))=n-r,\ \dim(\text{null}(A^T))=m-r$$

Enlarge the previous equivalence theorem

If $A \in \mathbb{F}^{n\times n}$ in which there are no duplicate rows and no duplicate columns, then the following statements are equivalent.

(a) A is invertible

(b) $A\mathbf{x} = 0$ has only the trivial solution

(c) The reduced row echelon form of A is I_n

(d) A is expressible as a product of elementary matrices.

(e) $A\mathbf{x} = b$ is consistent for every $n\times 1$ matrix b.

(f) $A\mathbf{x} = b$ has exactly one solution for every $n\times 1$ matrix b.

(g) $\det(A) \neq 0$

(h) The column vectors of A are linearly independent

(i) The row vectors of A are linearly independent

(j) The column vectors of A span $\mathbb{R}^n$

(k) The row vectors of A span $\mathbb{R}^n$

(l) The column vectors of A form a basis for $\mathbb{R}^n$

(m) The row vectors of A form a basis for $\mathbb{R}^n$

(n) $\text{rank}(A) = n$

(o) $\text{nullity}(A) = 0$