Eigenvalues, eigenvectors, and diagonalization

Definition If A is an $n \times n$ matrix, then if $A\mathbf{x} = \lambda \mathbf{x}$ for some scalar λ and a nonzero vector $\mathbf{x} \in \mathbb{R}^n$, then

$$\begin{cases} \lambda \text{ is called an eigenvalue of } A \\ \mathbf{x} \text{ is called an eigenvector corresponding to } \lambda \end{cases}$$

Example Given $A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$, then $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\lambda = 3$,

$$A\mathbf{x} = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} = 3 \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 3\mathbf{x}$$

Computing Eigenvalues and Eigenvectors

$$A\mathbf{x} = \lambda \mathbf{x}$$

$$\Leftrightarrow A\mathbf{x} = \lambda \cdot I \mathbf{x}$$

$$\Leftrightarrow (A - \lambda I)\mathbf{x} = \mathbf{0}$$

We want to have nonzero solution for $\mathbf{x}$

Theorem If A is an $n \times n$ matrix, then λ is an eigenvalue of A if and only if $\underline{\det(A - \lambda I)} = 0$

↳ characteristic polynomial

characteristic equation

Example: $A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$, $\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 0 \\ 8 & -1-\lambda \end{vmatrix}$

$$\det(A - \lambda I) = 0 \Rightarrow (\lambda - 3)(\lambda + 1) = 0$$

$\therefore$ eigenvalues of A are $\lambda = 3$, $\lambda = -1$

We can expand the characteristic equation as

$$\lambda^n + c_1\lambda^{n-1} + \cdots + c_n = 0$$

Say $P(\lambda) = \lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \cdots + c_n$

the characteristic polynomial of A.

Example Find the eigenvalues of $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{bmatrix}$

(sol): The characteristic polynomial of A:

$$\det(A - \lambda I) = \det \begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 4 & -17 & 8-\lambda \end{bmatrix}$$

$$= -\lambda^3 + 8\lambda^2 - 17\lambda + 4$$

$\det(A - \lambda I) = 0$

$\Rightarrow 1\cdot\lambda^3 - 8\lambda^2 + 17\lambda - 4 = 0$

divisors: ± 1

divisors: $\pm 1, \pm 2, \pm 4$

First, we found $\lambda = 4$ is an integer solution

$\therefore (\lambda - 4)(\lambda^2 - 4\lambda + 1) = 0$

$\Rightarrow \lambda = 4,\ \lambda = 2 \pm \sqrt{3}$, we found three eigenvalues.

Example $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix}$. Find A's eigenvalues.

$$\det(A-\lambda I) = \det \begin{bmatrix} a_{11}-\lambda & a_{12} & a_{13} & a_{14} \\ 0 & a_{22}-\lambda & a_{23} & a_{24} \\ 0 & 0 & a_{33}-\lambda & a_{34} \\ 0 & 0 & 0 & a_{44}-\lambda \end{bmatrix}$$

$$= (a_{11}-\lambda)(a_{22}-\lambda)(a_{33}-\lambda)(a_{44}-\lambda)$$

$\therefore$ The characteristic equation is

$$(\lambda - a_{11})(\lambda - a_{22})(\lambda - a_{33})(\lambda - a_{44}) = 0$$

$$\Rightarrow \lambda = a_{11},\ \lambda = a_{22},\ \lambda = a_{33},\ \lambda = a_{44}$$

Theorem If A is an $n \times n$ triangular matrix, then its eigenvalues are its diagonal entries.

Remark If A is an $n \times n$ matrix, then

(a) λ is an eigenvalue of A
(b) λ is a solution of $\det(A-\lambda I) = 0$
(c) $(A-\lambda I)\mathbf{x} = 0$ has nontrivial solutions.
(d) There is a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$

$\Leftarrow$ equivalent!

§ Finding Eigenvectors and Bases for Eigenspaces

For an eigenvalue λ, the eigenvectors of A corresponding to λ are the nonzero vectors satisfying

$$(A-\lambda I)\mathbf{x} = 0$$

We call its "solution space" eigenspace of A corresponding to λ.

$\Leftrightarrow$ $\begin{cases} \text{null}(A-\lambda I) \\ \text{set of vectors for which } A\mathbf{x} = \lambda\mathbf{x} \end{cases}$

<u>Example</u> Find bases for the eigenspaces of $A=\begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix}$

(sol): The characteristic equation of A:

$$\begin{vmatrix} -1-\lambda & 3 \\ 2 & -\lambda \end{vmatrix} = \lambda(\lambda+1)-6 = 0$$

$$\Rightarrow (\lambda-2)(\lambda+3)=0$$

$\therefore$ eigenvalues of A : $\lambda=2$, $\lambda=-3$

$\therefore$ There are two eigenspaces of A.

Let $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $(A-\lambda I)\mathbf{x} = 0 \Rightarrow \begin{bmatrix} -1-\lambda & 3 \\ 2 & -\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

(1) $\lambda=2$: $\begin{bmatrix} -3 & 3 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \Rightarrow x_1 = t,\ x_2 = t,\ t\in\mathbb{R}$

$\therefore \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix},\ t\in\mathbb{R}$

(2) $\lambda=-3$: $\begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\frac{3}{2}t \\ t \end{bmatrix} = t\begin{bmatrix} -\frac{3}{2} \\ 1 \end{bmatrix},\ t\in\mathbb{R}$

$\therefore \left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$ is a basis for the eigenspace corresponding to $\lambda=2$

$\left\{\begin{bmatrix} -3/2 \\ 1 \end{bmatrix}\right\}$ $\lambda=-3$ #

Example. Find bases for the eigenspaces of $A=\begin{bmatrix}0&0&-2\\1&2&1\\1&0&3\end{bmatrix}$

sol): $\det(A-\lambda I)=0$

$\Rightarrow \lambda^3-5\lambda^2+8\lambda-4=0$

$\Rightarrow (\lambda-1)(\lambda-2)^2=0$

$\therefore$ eigenvalues : $1, 2$, Let $\mathbf{x}=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$,

for $\lambda=1$,

$$\begin{bmatrix}-1&0&-2\\1&1&1\\1&0&2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}-2s\\s\\s\end{bmatrix}=s\begin{bmatrix}-2\\1\\1\end{bmatrix},\quad s\in\mathbb{R}$$

for $\lambda=2$:

$$\begin{bmatrix}-2&0&-2\\1&0&1\\1&0&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}-s\\t\\s\end{bmatrix}=s\begin{bmatrix}-1\\0\\1\end{bmatrix}+t\begin{bmatrix}0\\1\\0\end{bmatrix},\quad s,t\in\mathbb{R}$$

linearly independent (Check!)

$\therefore \left\{\begin{bmatrix}-2\\1\\1\end{bmatrix}\right\}$ is a basis for the eigenspace corresponding to $\lambda=1$

$\left\{\begin{bmatrix}-1\\0\\1\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$ is a basis for the eigenspace corresponding to $\lambda=2$

Theorem : A square matrix A is invertible if and only if $\lambda = 0$ is NOT an eigenvalue of A.

(proof sketch):

characteristic equation :

$$\lambda^n + c_1 \lambda^{n-1} + \cdots + c_n = 0 \quad \cdots (*)$$

$\therefore$ $\lambda = 0$ is a solution of $(*)$ $\Leftrightarrow$ $c_n = 0$

note that $\det(A - \lambda I) = 0$ for $\lambda = 0$

$$\Leftrightarrow \det(A) = c_n = 0$$

$\therefore$ A is invertible $\Leftrightarrow$ $c_n \neq 0$

§ Diagonalization

Suppose we have matrices $A, P \in \mathbb{F}^{n\times n}$ and P is invertible

Consider the transformation of A:

$$A \longrightarrow \underbrace{P^{-1}AP}_{= B}$$

We know that $\det(B) = \det(P^{-1}AP)$

$$= \det(P^{-1})\det(A)\det(P)$$

invariant

$$= \frac{1}{\det(P)}\det(A)\det(P)$$

$$= \det(A)$$

Definition (Similarity)

If A and B are square matrices, then we say that B is similar to A if there exists an invertible matrix P such that $B = P^{-1}AP$

Note: B is similar to $A \Rightarrow A$ is similar to B

$$B = P^{-1}AP \Rightarrow A = PBP^{-1} = (P^{-1})^{-1}B(P^{-1})$$

Definition (Diagonalizable)

A square matrix A is diagonalizable if it is similar to some diagonal matrix.

If $P^{-1}AP = B$, B is a diagonal matrix, then we say P diagonalize A.

5.2.1

Theorem If A is an $n \times n$ matrix, then the following statements are equivalent:

(a) A is diagonalizable.

(b) A has n linearly independent eigenvectors.

$(a) \Rightarrow (b)$:

$\because$ A is diagonalizable

$\exists$ invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$

$\Leftrightarrow$ $AP = PD$

Let $p_1, p_2, \cdots, p_n$ be the column vectors of P and assume that $D = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 \ddots & \\ 0 & & \lambda_n \end{bmatrix}$

$\therefore AP = A[p_1\ p_2 \cdots p_n] = [Ap_1\ Ap_2 \cdots Ap_n]$

also $PD = [p_1\ p_2 \cdots p_n] \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$

$= [\lambda_1 p_1\ \lambda_2 p_2 \cdots \lambda_n p_n]$

$\therefore Ap_1 = \lambda_1 p_1,\ Ap_2 = \lambda_2 p_2,\ \cdots,\ Ap_n = \lambda_n p_n$

$\because P$ is invertible $\therefore p_1, p_2, \cdots, p_n$ are linearly independent column vectors

$\Rightarrow p_1, p_2, \cdots, p_n$ are n linearly independent eigenvectors.

$(b) \Rightarrow (a)$: Let $p_1, p_2, \cdots, p_n$ be the n linearly independent eigenvectors of A and $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the corresponding eigenvalues

Let $P = [p_1\ p_2 \cdots p_n]$ and $D = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 \ddots & \\ 0 & & \lambda_n \end{bmatrix}$

then $AP = A[p_1 p_2 \cdots p_n] = [Ap_1\ Ap_2 \cdots Ap_n]$

$= [\lambda_1 p_1\ \lambda_2 p_2 \cdots \lambda_n p_n] = PD$

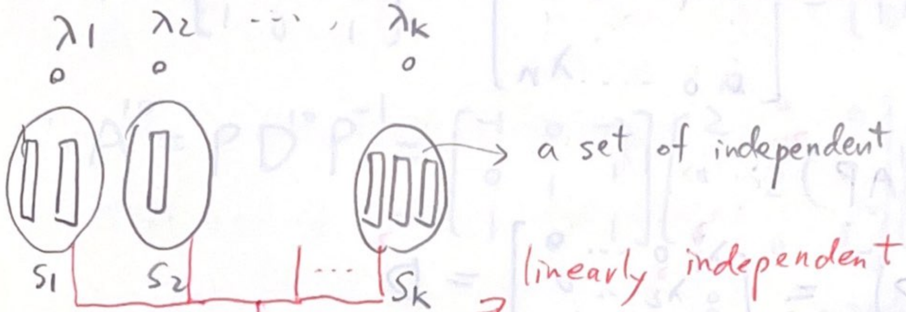
$\because p_1, p_2, \cdots, p_n$ are linearly independent

$\therefore P$ is invertible $\Rightarrow P^{-1}AP = D \Rightarrow A$ is diagonalizable #

5.2.2 (1)

Theorem If $\lambda_1, \lambda_2, \cdots, \lambda_k$ are distinct eigenvalues of matrix A, and if $v_1, v_2, \cdots, v_k$ are the corresponding eigenvectors, then $\{v_1, v_2, \cdots, v_k\}$ is a linearly independent set.

(2) An $n \times n$ matrix with n distinct eigenvalues is diagonalizable



Let's Find a Matrix P that Diagonalizes a Matrix A!

Example Find a matrix P that diagonalizes

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

(sol): $\det(A - \lambda I) = 0 \Rightarrow (\lambda - 1)(\lambda - 2)^2 = 0$

For $\lambda = 2$: We found the bases: $P_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, $P_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

For $\lambda = 1$: We found the basis: $P_3 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$

$\therefore$ Let $P = [P_1 \; P_2 \; P_3] = \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

Find P^{-1}: $P^{-1} = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ -1 & 0 & -1 \end{bmatrix}$

Then

$$P^{-1}AP = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

If we take $P = \begin{bmatrix} -1 & -2 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$, then $P^{-1}AP = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

§ Computing Powers of a Matrix!

Suppose that A is diagonalizable $n \times n$ matrix that can be diagonalized by P.

Then, $P^{-1}AP = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} = D$

Consider $(P^{-1}AP)^2$:

$$(P^{-1}AP)^2 = \begin{bmatrix} \lambda_1^2 & 0 & \cdots & 0 \\ 0 & \lambda_2^2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n^2 \end{bmatrix} = D^2$$

Actually,

$$(P^{-1}AP)^2 = (P^{-1}AP)(P^{-1}AP) = P^{-1}APP^{-1}AP$$
$$= P^{-1}AI_nAP = P^{-1}AAP = P^{-1}A^2P$$

We can generalize this to:

$$(P^{-1}AP)^k = P^{-1}A^kP$$

$$\therefore (P^{-1}AP)^k = P^{-1}A^kP = D^k = \begin{bmatrix} \lambda_1^k & & 0 \\ & \lambda_2^k & \\ 0 & & \ddots \ \lambda_n^k \end{bmatrix}$$

Also,

$$A^k = PD^kP^{-1} = P\begin{bmatrix} \lambda_1^k & & 0 \\ & \lambda_2^k & \\ 0 & & \ddots \ \lambda_n^k \end{bmatrix}P^{-1}$$

Example : Let $A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$, find A^{10}

(sol) : Recall from the previous example,

$$P = \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \text{ and } P^{-1}AP = D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore A^{10} = PD^{10}P^{-1} = \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2^{10} & 0 & 0 \\ 0 & 2^{10} & 0 \\ 0 & 0 & 1^{10} \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ -1 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -1022 & 0 & -2046 \\ 1023 & 1024 & 1023 \\ 1023 & 0 & 2047 \end{bmatrix}$$

Example (Non-diagonalizable)

Show that $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -3 & 5 & 2 \end{bmatrix}$ is NOT diagonalizable.

sol):

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 1 & 2-\lambda & 0 \\ -3 & 5 & 2-\lambda \end{vmatrix} = -(\lambda-1)(\lambda-2)^2$$

$$\det(A - \lambda I) = 0 \Rightarrow (\lambda-1)(\lambda-2)^2 = 0$$

$\therefore$ eigenvalues : $\lambda = 1,\ \lambda = 2$

For $\lambda = 1$: $P_1 = \begin{bmatrix} \frac{1}{8} \\ -\frac{1}{8} \\ 1 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ -3 & 5 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

For $\lambda = 2$: $P_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ $\begin{bmatrix} -1 & 0 & 0 \\ 1 & 0 & 0 \\ -3 & 5 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

Since A is an 3×3 matrix and there are only two basis vectors

$\therefore$ A is NOT diagonalizable $\#$

(sol 2) : NOTE : The problem only asks you to determine WHETHER a matrix is diagonalizable.

$\Rightarrow$ We can find the $\boxed{\text{dimensions}}$ of the eigenspaces.

For $\lambda = 1$:

$$\boxed{\begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ -3 & 5 & 1 \end{bmatrix}}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\hookrightarrow$ rank : 2 , nullity : 1 $\Rightarrow$ the eigenspace corresponding to $\lambda = 1$ is ONE-dimensional

For $\lambda = 2$,

$$\boxed{\begin{bmatrix} -1 & 0 & 0 \\ 1 & 0 & 0 \\ -3 & 5 & 0 \end{bmatrix}}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\hookrightarrow$ rank: 2 nullity : 1 $\Rightarrow$ the eigenspace corresponding to $\lambda = 2$ is ONE-dimensional

$\Rightarrow$ TOTAL : $1 + 1 = 2 < 3 \Rightarrow$ NOT diagonalizable

Example : Determine whether or not A is diagonalizable.

$$A = \begin{bmatrix} -1 & 2 & 4 & 0 \\ 0 & 3 & 1 & 7 \\ 0 & 0 & 5 & 8 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

(sol): $\because$ A is triangular

$\therefore$ The eigenvalues of A are $-1, 3, 5, -2$

$\Rightarrow$ A has 4 distinct eigenvalues

$\therefore$ A is diagonalizable $\divideontimes$