Midterm Exam of Linear Algebra

15:10 – 17:00, 16 April 2024; Room E401

Note: Cell phones and any calculator are forbidden.

Part I: True (T) or False (F) (40%; each for 4%)

1. A homogeneous system of linear equations can have infinitely many solutions.
2. If A and B are invertible of the same size, then AB is invertible and $(AB)^{-1} = A^{-1}B^{-1}$.
3. If A is an invertible symmetric matrix, then A^{-1} is symmetric.
4. The product of elementary matrices is always invertible.
5. For every matrix $A \in \mathbb{R}^{n\times m}$ and $n, m \in \mathbb{N}$, we have $\operatorname{tr}(A^\top A) = \operatorname{tr}(AA^\top)$.
6. For any invertible and symmetric $n \times n$ matrix A, we have A^n is invertible and $(A^n)^{-1} = (A^\top)^{-n}$.
7. If $B = R_2^{(2)} R_{12}^{(3)} A$ where $R_2^{(2)}, R_{12}^{(3)}$ are two elementary matrices, then $A = R_{21}^{(-3)} R_2^{(1/2)} B$.
8. For $A \in \mathbb{R}^{2\times 2}, T(A) = A - A^\top$ is a linear transformation.
9. If $\mathbf{b}, \mathbf{x} \in \mathbb{R}^n$ and $\mathbf{b} \neq \mathbf{0}$, then $T(\mathbf{x}) = \mathbf{x} + \mathbf{b}$ is a matrix transformation on $\mathbb{R}^n$.
10. If $A^{n+1} = \mathbf{0}$, then $(I - A)^{-1} = I + A + A^2 + \cdots + A^n$.

Part II: Calculations. (80%; each for 8%; ONLY THE ANSWERS ARE REQUIRED)

1. Solve the system with variables x_1, x_2, x_3, x_4 : $\begin{bmatrix} 1 & 2 & 3 & 4 & 10 \\ 2 & 3 & 4 & 5 & 14 \\ 3 & 4 & 5 & 6 & 18 \\ 4 & 5 & 6 & 7 & 22 \end{bmatrix}$
2. Compute A^{10} where $A = \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}$
3. Compute the inverse of $A = \begin{bmatrix} 6 & 2 & 3 \\ 3 & 1 & 1 \\ 10 & 3 & 4 \end{bmatrix}$.
4. Let $A = \begin{bmatrix} -2 & 0 \\ 6 & 1 \end{bmatrix}$. Find elementary matrices E_1, E_2 and E_3 such that $A = E_1E_2E_3$.
5. Find the standard matrix A for the linear transformation $T : \mathbb{R}^3 \mapsto \mathbb{R}^3$ for which
   $T\left(\begin{bmatrix} -2 \\ 3 \\ -4 \end{bmatrix}\right) = \begin{bmatrix} 5 \\ 3 \\ 14 \end{bmatrix}, T\left(\begin{bmatrix} 3 \\ -2 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} -4 \\ 6 \\ -14 \end{bmatrix}$ and $T\left(\begin{bmatrix} -4 \\ -5 \\ 5 \end{bmatrix}\right) = \begin{bmatrix} -6 \\ -40 \\ -2 \end{bmatrix}$
6. Compute $\begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{2024}$.
7. Find the inverse of $\begin{bmatrix} \cos(-\theta) & \sin(-\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$.

8. Suppose that $A = \begin{bmatrix} 1 & 4 \\ -2 & 3 \\ 1 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix}$ and $C = \begin{bmatrix} 16 & 12 & -12 \\ 12 & -2 & 2 \\ -8 & 0 & 0 \end{bmatrix}$. Find a matrix K such that $AKB = C$.

9. Given the Fibonacci sequence $F_0 = 0, F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. Compute Q^n for which $Q = \begin{bmatrix} F_2 & F_1 \\ F_1 & F_0 \end{bmatrix}$. (Please give your answer of Q^n as a 2×2 matrix)

10. Compute the determinant of the following matrix:

$$\begin{bmatrix} 5 & 4 & 8 & 1 & 1 \\ 4 & 5 & 7 & 1 & 2 \\ 3 & 6 & 6 & 1 & 3 \\ 2 & 7 & 5 & 1 & 4 \\ 1 & 8 & 4 & 1 & 5 \end{bmatrix}.$$

(計算紙)