Mathematics for Machine Learning (2025 Fall)

Assignment 04*

Chuang-Chieh Lin

Department of Computer Science and Engineering,
National Taiwan Ocean University

1. (25%) Consider two D-dimensional Gaussians in **the same variable x**:

$$\mathcal{N}(\mathbf{x} \mid \mathbf{a}, \boldsymbol{A}) \text{ and } \mathcal{N}(\mathbf{x} \mid \mathbf{b}, \boldsymbol{B}).$$

Please prove that their product is proportional to another Gaussian:

$$\mathcal{N}(\mathbf{x} \mid \mathbf{a}, \boldsymbol{A})\, \mathcal{N}(\mathbf{x} \mid \mathbf{b}, \boldsymbol{B}) = \rho \mathcal{N}(\mathbf{x} \mid \mathbf{c}, \boldsymbol{C}),$$

where

$$\boldsymbol{C} = (\boldsymbol{A}^{-1} + \boldsymbol{B}^{-1})^{-1}, \quad \mathbf{c} = \boldsymbol{C}(\boldsymbol{A}^{-1}\mathbf{a} + \boldsymbol{B}^{-1}\mathbf{b}),$$

and the scaling constant ρ is

$$\rho = (2\pi)^{-D/2} \det(\boldsymbol{A} + \boldsymbol{B})^{-1/2} \exp\left(-\tfrac{1}{2}(\mathbf{a} - \mathbf{b})^\top (\boldsymbol{A} + \boldsymbol{B})^{-1}(\mathbf{a} - \mathbf{b})\right).$$

2. (25%) Consider the problem:

$$\min_{\mathbf{x} \in \mathbb{R}^d} \mathbf{c}^\top \mathbf{x}$$
$$\text{subject to } \mathbf{A}\mathbf{x} \preceq \mathbf{b} \text{ for } \mathbf{A} \in \mathbb{R}^{m \times d}, \mathbf{b} \in \mathbb{R}^m, \text{ and } \mathbf{c} \in \mathbb{R}^d.$$

   (a) Please provide its Lagrangian function $\mathcal{L}(\mathbf{x}, \lambda)$.
   (b) Please list the dual problem.

3. (25%) Given $f : \mathbb{R}^+ \to \mathbb{R}$; $f(x) = x \ln x$. Compute $f(t) + (\nabla_x f)(t)^\top (z - t)$ for $t = e^2, z = e^3$.

4. (25%) Let X be a continuous random variable with pdf $f_X : [0, 1] \to [0, 1]$: $f_X(x) = 4x^3$. Compute the pdf of $Y = X^2$.

* List the required intermediate steps next to each problem. Note that any answers generated directly by AI are invalid for this assignment.