Assignment 1
Due date: 3 October 2023
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1. (30%) Prove or disprove the following sets are subspaces of $\mathbb{R}^3$.
   a. $A = \{(\lambda, \lambda + \mu^3, \lambda - \mu^3) \mid \lambda, \mu \in \mathbb{R}\}$. (10%)
   b. $B = \{(\lambda^2, -\lambda^2, 0) \mid \lambda \in \mathbb{R}\}$. (10%)
   c. $C = \{(\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 \mid \xi_2 \in \mathbb{Z}\}$. (10%)

2. (20%) Determine whether the following sets of vectors are linearly independent.
   a. $\mathbf{x}_1 = \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}, \ \mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \ \mathbf{x}_3 = \begin{bmatrix} 3 \\ -3 \\ 8 \end{bmatrix}$. (10%)
   b. $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \ \mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \ \mathbf{x}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$. (10%)

3. (25%) Write

$$\mathbf{y} = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$$

   as a linear combination of

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \ \mathbf{x}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \ \mathbf{x}_3 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}.$$

4. (25%) Consider two subspaces of $\mathbb{R}^4$:

$$U_1 = \operatorname{span}\left(\begin{bmatrix} 1 \\ 1 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right), \quad U_2 = \operatorname{span}\left(\begin{bmatrix} -1 \\ -2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 6 \\ -2 \\ -1 \end{bmatrix} \right).$$

   Determine a basis of $U_1 \cap U_2$.