Midterm Exam of MML
10:10 – 12:00, 7 November 2023; Room E416
Note: Cell phones and any calculator are forbidden.

Part I: True (T) or False (F) (65%; each for 5%)

1. Every invertible matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ is diagonalizable.
2. $0 \in \mathbb{R}$ can be an eigenvalue of a square matrix.
3. Both $\mathbf{A}\mathbf{A}^\top$ and $\mathbf{A}^\top\mathbf{A}$ are symmetric. ($\mathbf{A} \in \mathbb{R}^{m\times n}$, $m, n \in \mathbb{N}$.)
4. Any symmetric matrix is symmetric to a diagonal matrix.
5. For any $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{A} \in \mathbb{R}^{n\times n}$, we have $\mathbf{x}^\top\mathbf{A}\mathbf{x} = \mathrm{tr}(\mathbf{A}\mathbf{x}\mathbf{x}^\top)$.
6. Every symmetric matrix is positive semidefinite.
7. For any $\mathbf{A} \in \mathbb{R}^{m\times n}$, $\mathbf{A}\mathbf{A}^\top$ and $\mathbf{A}^\top\mathbf{A}$ have the same eigenvalues.
8. If $\mathbf{A} \in \mathbb{R}^{n\times n}$ consists of n orthonormal nonzero column vectors, then $\mathbf{A}^{-1} = \mathbf{A}^\top$.
9. $S = \{(\lambda^2, -\lambda^2, 0) \mid \lambda \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^3$.
10. For $A \in \mathbb{R}^{2\times 2}$, $T(A) = A - A^\top$ is a linear transformation.
11. If $\mathbf{b}$ is a nonzero vector in $\mathbb{R}^n$, then $T(\mathbf{x}) = \mathbf{x} + \mathbf{b}$ is a linear transformation on $\mathbb{R}^n$.
12. $\mathbf{A}\mathbf{A}^\top$ always has nonnegative eigenvalues. ($\mathbf{A} \in \mathbb{R}^{m\times n}$, $m, n \in \mathbb{N}$.)
13. For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$, $f(\mathbf{x}, \mathbf{y}) := x_1y_1 - (x_1y_2 + x_2y_1) - 2x_2y_2$ is an inner product.

Part II: Calculations. (65%; each for 5%; ONLY THE ANSWERS ARE REQUIRED)

1. (5%) Consider a transformation matrix $\mathbf{A}_\Phi = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ of a linear mapping $\Phi : \mathbb{R}^2 \mapsto \mathbb{R}^2$ with respect to the standard basis. Let $B = ([-1,\ 1]^\top, [1,\ 1]^\top)$ be another basis of $\mathbb{R}^2$. Please compute the transformation matrix $\tilde{\mathbf{A}}_\Phi$ with respect to B.

2. (5%) Find the transformation matrix $\mathbf{A}_T$ for the linear transformation $T : \mathbb{R}^3 \mapsto \mathbb{R}^3$ for which
$T\left(\begin{bmatrix} -2 \\ 3 \\ -4 \end{bmatrix}\right) = \begin{bmatrix} 5 \\ 3 \\ 14 \end{bmatrix}$, $T\left(\begin{bmatrix} 3 \\ -2 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} -4 \\ 6 \\ -14 \end{bmatrix}$ and $T\left(\begin{bmatrix} -4 \\ -5 \\ 5 \end{bmatrix}\right) = \begin{bmatrix} -6 \\ -40 \\ -2 \end{bmatrix}$

3. (5%) Find a Cholesky Factorization of $\begin{bmatrix} 4 & 8 \\ 8 & 20 \end{bmatrix}$.

4. (5%) Diagonalize $\begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix} = \mathbf{P}\mathbf{D}\mathbf{P}^\top$ such that $\mathbf{P}$ consists of orthonormal column vectors.

5. (10%) Given $\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$.

   a. Find a singular value decomposition for $\mathbf{A}$.

b. Let $\|\mathbf{A}\|_2 = \max_{\mathbf{x}} \frac{\|\mathbf{A}\mathbf{x}\|_2}{\|\mathbf{x}\|_2}$ and $\hat{\mathbf{A}}_1$ be the rank-1 approximation of $\mathbf{A}$, compute $\|\mathbf{A} - \hat{\mathbf{A}}_1\|_2 =$ _____.

6. (10%) Given $f(\mathbf{x}) = \mathbf{x}\mathbf{x}^\top$ where $\mathbf{x} \in \mathbb{R}^n$.

   a. What is the shape (i.e., dimensions) of $\frac{d}{d\mathbf{x}} f(\mathbf{x})$?

   b. What is $\frac{d}{dx_1} f(\mathbf{x})$?

7. (5%) Compute $\frac{d}{d\mathbf{x}} f(\mathbf{x}, \mathbf{y})$, where $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $f(\mathbf{x}, \mathbf{y}) = \mathbf{x}^\top \mathbf{y}$.

8. (5%) Given the formula $\frac{\partial \mathbf{x}^\top \mathbf{B}\mathbf{x}}{\partial \mathbf{x}} = \mathbf{x}^\top(\mathbf{B} + \mathbf{B}^\top)$ for a square matrix $\mathbf{B}$, compute the gradient $\frac{\partial}{\partial \mathbf{s}} \big((\mathbf{x} - \mathbf{A}\mathbf{s})^\top \mathbf{A}\mathbf{A}^\top (\mathbf{x} - \mathbf{A}\mathbf{s}) + \|\mathbf{s}\|^2\big)$.

9. (5%) Compute the derivatives $df/d\mathbf{x}$, where $f(z) = \ln(1+z)$, and $z = \mathbf{x}^\top \mathbf{x}$, for $\mathbf{x} \in \mathbb{R}^D$.

10. (5%) Compute the determinant of the matrix $\begin{bmatrix} 2 & 0 & 1 & 2 & 0 & 1 \\ 2 & -1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 & 2 & 1 \\ -2 & 0 & 2 & -1 & 2 & 0 \\ 2 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$.

11. (5%) Given $T_{\mathbf{A}} : \mathbb{R}^5 \mapsto \mathbb{R}^3$, where $\mathbf{A} = \begin{bmatrix} 1 & 2 & -1 & -1 & -1 \\ 2 & -1 & 1 & 2 & -2 \\ 3 & -4 & 3 & 5 & 3 \end{bmatrix}$,

    $\mathrm{rank}(\mathbf{A}) + \ker(T_{\mathbf{A}}) =$ _____.