Online Learning

— Online Mirror Descent

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Credits for the resource

The slides are based on the lectures of Prof. Luca Trevisan:
https://lucatrevisan.github.io/40391/index.html

the lectures of Prof. Shipra Agrawal:
https://ieor8100.github.io/mab/

the lectures of Prof. Francesco Orabona:
https://parameterfree.com/lecture-notes-on-online-learning/
the monograph: https://arxiv.org/abs/1912.13213

and also Elad Hazan's textbook:
Introduction to Online Convex Optimization, 2nd Edition.

I would like to especially thank Prof. Francesco Orabona for the discussion with me about the details for this part of lectures.

Outline

1. Uninformative Subgradients
2. Reinterpreting the Online Subgradient Descent
3. An Alternative Distance Measure: Bregman Divergence
4. Online Mirror Descent - The First Attempt
5. The Mirror Interpretation

Outline

Online Subgradient Descent (OSD)

- Consider the simplified case that $f_t(\cdot) = f(\cdot)$ for all $t > 0$.
- The key property for the convergence of OSD:

$$f(\mathbf{x}_t) - f(\mathbf{u}) \leq \langle \mathbf{g}_t, \mathbf{x}_t - \mathbf{u} \rangle, \ \forall \mathbf{u}.$$

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- However, for $\mathbf{x} \in \mathbb{R}^2$, consider the following two functions:
  - $f(\mathbf{x}) = \max\{-x_1, x_1 - x_2, x_1 + x_2\}$.
  - $f(\mathbf{x}) = \max\{x_1^2 + (x_2 + 1)^2, x_1^2 + (x_2 - 1)^2\}$.

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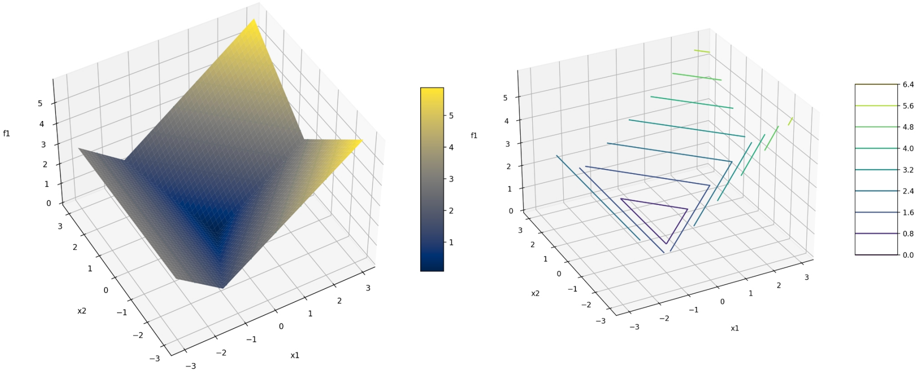
- Moving toward the direction of the negative subgradient may not decrease the objective (loss).

Uninformative Subgradients



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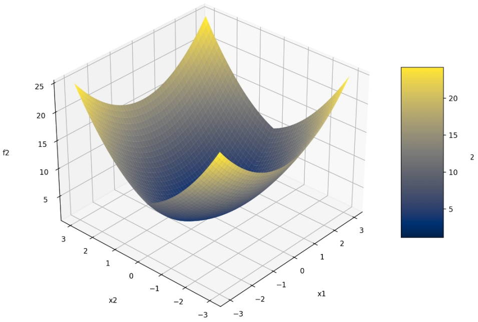

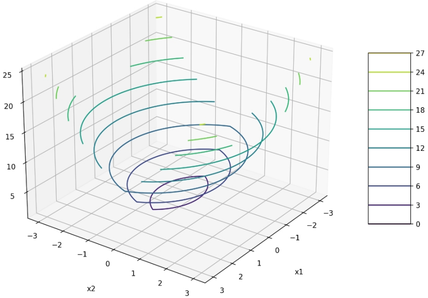

Outline

A Linear Lower Bound by a Subgradient

- We can have a linear lower bound on function f around $\mathbf{x}_0$:

$$f(\mathbf{x}) \geq \tilde{f}(\mathbf{x}) := f(\mathbf{x}_0) + \langle \mathbf{g}, \mathbf{x} - \mathbf{x_0} \rangle, \forall \mathbf{x} \in V.$$

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  - Let's say $V \subseteq \mathbb{R}^d$ is the domain.
- Note that over unbounded domains the minimizer of linear function at the right-hand side above is $-\infty$.

A Principle of Moderation

- Minimizing the previous lower bound only in a neighborhood of $\mathbf{x}_t$.

$$\mathbf{x}_{t+1} = \arg\min_{\mathbf{x}\in V} f(\mathbf{x}_t) + \langle \mathbf{g}, \mathbf{x} - \mathbf{x}_t \rangle$$
$$\text{subject to } \|\mathbf{x}_t - \mathbf{x}\|^2 \leq h,$$
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- Unconstrained formulation: (assume $\eta > 0$)

$$\arg\min_{\mathbf{x}\in V} f(\mathbf{x}_t) + \langle \mathbf{g}, \mathbf{x} - \mathbf{x}_t \rangle + \frac{1}{2\eta}\|\mathbf{x}_t - \mathbf{x}\|_2^2.$$

Solving the minimization (ignore non-variable terms):

$$\arg\min_{\mathbf{x}\in V}\langle \mathbf{g}_t, \mathbf{x}\rangle + \frac{1}{2\eta_t}\|\mathbf{x}_t - \mathbf{x}\|_2^2 = \arg\min_{\mathbf{x}\in V} 2\eta_t\langle \mathbf{g}_t, \mathbf{x}\rangle + \|\mathbf{x}_t - \mathbf{x}\|_2^2$$

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= \quad & \arg\min_{\mathbf{x}\in V} \|\mathbf{x}_t - \eta_t \mathbf{g}_t - \mathbf{x}\|_2^2 \\
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where $\Pi_V(\mathbf{x}) = \arg\min_{\mathbf{y}\in V} \|\mathbf{x} - \mathbf{y}\|_2$ (Euclidean projection onto V).

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- So, we rediscovered the online subgradient descent with projection!

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$$\underset{\mathbf{x}\in V}{\arg\min}\langle \mathbf{g}_t, \mathbf{x}\rangle + \frac{1}{2\eta_t}\|\mathbf{x}_t - \mathbf{x}\|_2^2$$

$$\Rightarrow \quad \underset{\mathbf{x}\in V}{\arg\min}\langle \mathbf{g}_t, \mathbf{x}\rangle + \frac{1}{\eta_t}B_\psi(\mathbf{x}; \mathbf{x}_t)$$

Note: When $\psi(\mathbf{x}) = \frac{1}{2}\|\mathbf{x}\|_2^2$, the two updates are exactly the same.

Outline

Strictly Convexity

Strictly Convex Functions

A function $f : V \subseteq \mathbb{R}^d \mapsto \mathbb{R}$, where V is a convex set, is strictly convex if

$$f(\alpha \mathbf{x} + (1-\alpha)\mathbf{y}) < \alpha f(\mathbf{x}) + (1-\alpha) f(\mathbf{y}),$$

$\forall \mathbf{x}, \mathbf{y} \in V, \mathbf{x} \neq \mathbf{y}, \alpha \in (0,1)$.

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- Strong convexity w.r.t. any norm implies strict convexity.
- If f is differentiable, strict convexity implies that

$$f(\mathbf{y}) > f(\mathbf{x}) + \langle \nabla f(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle$$

for $\mathbf{x} \neq \mathbf{y}$.

Bregman Divergence

Bregman Divergence

Let $\psi : X \mapsto \mathbb{R}$ be strictly convex and continuously differentiable on $\text{int}(X)$. The Bregman Divergence w.r.t. ψ is $B_\psi : X \times \text{int}(X) \mapsto \mathbb{R}$ defined as

$$B_\psi(\mathbf{x}; \mathbf{y}) = \psi(\mathbf{x}) - \psi(\mathbf{y}) - \langle \nabla\psi(\mathbf{y}), \mathbf{x} - \mathbf{y} \rangle.$$

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- $\psi(\mathbf{x}) \geq \psi(\mathbf{y}) + \langle \nabla\psi(\mathbf{y}), \mathbf{x} - \mathbf{y} \rangle, \forall \mathbf{y} \in X$, and equality holds only for $\mathbf{y} = \mathbf{x}$ ($\because$ strictly convexity of ψ).
- It can be a distance measure, though it is NOT Symmetric.

Examples (1/5)

- Consider a twice differentiable ψ in a ball B around $\mathbf{y}$ and $\mathbf{x} \in B$.
- By Taylor's theorem, there exists $0 \leq \alpha \leq 1$ such that

$$B_\psi(\mathbf{x};\mathbf{y}) = \psi(\mathbf{x}) - \psi(\mathbf{y}) - \nabla\psi(\mathbf{y})^\top(\mathbf{x}-\mathbf{y})$$

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for $\mathbf{z} = \alpha\mathbf{x} + (1 - \alpha)\mathbf{y}$.

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for $\mathbf{z} = \alpha\mathbf{x} + (1-\alpha)\mathbf{y}$.

⋆ A squared local norm depending on the Hessian of ψ.

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Examples (2/5)

- If ψ is λ-**strongly convex** w.r.t. a norm $\|\cdot\|$ in $\mathrm{int}(X)$, we have $B_\psi(\mathbf{x};\mathbf{y}) \geq \frac{\lambda}{2}\|\mathbf{x}-\mathbf{y}\|^2$.

Examples (3/5)

- If $\psi(\mathbf{x}) = \frac{1}{2}\|\mathbf{x}\|_2^2$, then

$$B_\psi(\mathbf{x}; \mathbf{y}) = \frac{1}{2}\|\mathbf{x}\|_2^2 - \frac{1}{2}\|\mathbf{y}\|_2^2 -$$

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Examples (4/5): **Exercise**

Please show that:

- If $\psi(\mathbf{x}) = \sum_{i=1}^{d} x_i \ln x_i$, and $X = \{\mathbf{x} \mid x_i \geq 0, \|\mathbf{x}\|_1 = 1\}$, then

$$B_\psi(\mathbf{x}; \mathbf{y}) = \sum_{i=1}^{d} x_i \ln \frac{x_i}{y_i}.$$

Examples (4/5): **Exercise**

Please show that:

- If $\psi(\mathbf{x}) = \sum_{i=1}^{d} x_i \ln x_i$, and $X = \{\mathbf{x} \mid x_i \geq 0, \|\mathbf{x}\|_1 = 1\}$, then

$$B_\psi(\mathbf{x}; \mathbf{y}) = \sum_{i=1}^{d} x_i \ln \frac{x_i}{y_i}.$$

⋆ This is the Kullback-Leibler divergence (KL-divergence) between two distributions $\mathbf{x}$ and $\mathbf{y}$.

Examples (5/5): **Exercise**

Please prove the following lemma.

Lemma [Chen & Teboulle 1993]

Let B_ψ be the Bregman divergence w.r.t. $\psi : X \mapsto \mathbb{R}$. Then, for any three points $\mathbf{x}, \mathbf{y} \in \mathrm{int}(X)$ and $\mathbf{z} \in X$, we have

$$B_\psi(\mathbf{z}; \mathbf{x}) + B_\psi(\mathbf{x}; \mathbf{y}) - B_\psi(\mathbf{z}; \mathbf{y}) = \langle \nabla\psi(\mathbf{y}) - \nabla\psi(\mathbf{x}), \mathbf{z} - \mathbf{x} \rangle.$$

Outline

Algorithm OMD

Input: Non-empty closed convex $V \subseteq X \subseteq \mathbb{R}^d$,
$\psi : X \mapsto \mathbb{R}$ strictly convex and continuously differentiable on $\text{int}(X)$,
$\mathbf{x}_1 \in V$ s.t. ψ is differentiable in $\mathbf{x}_1$,
$\eta_1, \ldots, \eta_T > 0$.

1: **for** $t \leftarrow 1$ to T **do**
2: Output $\mathbf{x}_t$
3: Receive $f_t : \mathbb{R}^d \mapsto (-\infty, +\infty]$ and suffer $f_t(\mathbf{x}_t)$
4: Set $\mathbf{g}_t \in \partial f_t(\mathbf{x}_t)$
5: $\mathbf{x}_{t+1} \leftarrow \arg\min_{\mathbf{x} \in V} \langle \mathbf{g}_t, \mathbf{x} \rangle + \frac{1}{\eta_t} B_\psi(\mathbf{x}; \mathbf{x}_t)$
6: **end for**

Fix Some Minor Issues

Add one of the following boundary conditions.

- $\lim_{\lambda \to 0} \langle \nabla \psi(\mathbf{x} + \lambda(\mathbf{y} - \mathbf{x})), \mathbf{y} - \mathbf{x} \rangle = -\infty$, for any $\mathbf{x} \in \text{boundary}(X)$, $\mathbf{y} \in \text{int}(X)$.
- $V \subseteq \text{int}(X)$.

When $\arg\min$ exists, $\mathbf{x}_{t+1} \in \mathrm{int}(X)$

Theorem

Let

- B_ψ be the Bregman divergence w.r.t. $\psi : X \mapsto \mathbb{R}$.
- $V \subseteq X$ be a non-empty closed and convex set.

Assume that previous two boundary conditions holds and the $\arg\min$ of the algorithm exists on all rounds, then we have $\mathbf{x}_{t+1} \in \mathrm{int}(X)$.

Existence of the arg min's

Theorem

Let

- $\lambda > 0$
- $f : \mathbb{R} \mapsto (-\infty, +\infty]$ a closed and λ-strongly convex w.r.t. $\|\cdot\|$.

Assume that $\text{dom}(\partial f) \neq \emptyset$. Then, f has exactly one minimizer.

Main Lemma

Lemma (Regret Inequality for OMD)

- ψ: λ-strongly convex w.r.t. $\|\cdot\|$ in V.
- B_ψ: the Bregman divergence w.r.t. $\psi : X \mapsto \mathbb{R}$.
- $V \subseteq X$: non-empty, closed & convex.
- Set $\mathbf{g}_t \in \partial f_t(\mathbf{x}_t)$.
- Assume one of the two boundary conditions holds.

Then for each $\mathbf{u} \in V$ and Algorithm OMD, we have

$$\eta_t(f_t(\mathbf{x}_t) - f_t(\mathbf{u})) \leq \eta_t\langle \mathbf{g}_t, \mathbf{x}_t - \mathbf{u}\rangle \leq B_\psi(\mathbf{u}; \mathbf{x}_t) - B_\psi(\mathbf{u}; \mathbf{x}_{t+1}) + \frac{\eta_t^2}{2\lambda}\|\mathbf{g}_t\|_*^2.$$

Proof of the Main Lemma (1/3)

Input: Non-empty closed convex $V \subseteq X \subseteq \mathbb{R}^d$,
$\psi : X \mapsto \mathbb{R}$ strictly convex and continuously differentiable on $\text{int}(X)$,
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$$\frac{\partial}{\partial \mathbf{x}} \left(\eta_t \langle \mathbf{g}_t, \mathbf{x} \rangle + B_\psi(\mathbf{x}; \mathbf{x}_t) \right)$$

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3: Receive $f_t : \mathbb{R}^d \mapsto (-\infty, +\infty]$ and suffer $f_t(\mathbf{x}_t)$
4: Set $\mathbf{g}_t \in \partial f_t(\mathbf{x}_t)$
5: $\mathbf{x}_{t+1} \leftarrow \arg\min_{\mathbf{x} \in V} \langle \mathbf{g}_t, \mathbf{x} \rangle + \frac{1}{\eta_t} B_\psi(\mathbf{x}; \mathbf{x}_t)$
6: **end for**

$$\frac{\partial}{\partial \mathbf{x}} \left(\eta_t \langle \mathbf{g}_t, \mathbf{x} \rangle + B_\psi(\mathbf{x}; \mathbf{x}_t) \right) = \eta_t \mathbf{g}_t + \nabla\psi(\mathbf{x}) - \nabla\psi(\mathbf{x}_t)$$

Proof of the Main Lemma (1/3)

Input: Non-empty closed convex $V \subseteq X \subseteq \mathbb{R}^d$,
$\psi : X \mapsto \mathbb{R}$ strictly convex and continuously differentiable on $\text{int}(X)$,
$\mathbf{x}_1 \in V$ s.t. ψ is differentiable in $\mathbf{x}_1$,
$\eta_1, \ldots, \eta_T > 0$.

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The optimality condition guarantees that

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The optimality condition guarantees that

$$\langle \eta_t \mathbf{g}_t + \nabla \psi(\mathbf{x}_{t+1}) - \nabla \psi(\mathbf{x}_t), \mathbf{u} - \mathbf{x}_{t+1} \rangle \geq 0, \ \forall \mathbf{u} \in V.$$

Proof of the Main Lemma (2/3)

$$\begin{aligned}
\langle \eta_t \mathbf{g}_t, \mathbf{x}_t - \mathbf{u} \rangle &= -\langle \eta_t \mathbf{g}_t + \nabla\psi(\mathbf{x}_{t+1}) - \nabla\psi(\mathbf{x}_t), \mathbf{u} - \mathbf{x}_{t+1} \rangle \\
&\quad + \langle \nabla\psi(\mathbf{x}_{t+1}) - \nabla\psi(\mathbf{x}_t), \mathbf{u} - \mathbf{x}_{t+1} \rangle + \langle \eta_t \mathbf{g}_t, \mathbf{x}_t - \mathbf{x}_{t+1} \rangle \\
&\leq \langle \nabla\psi(\mathbf{x}_{t+1}) - \nabla\psi(\mathbf{x}_t), \mathbf{u} - \mathbf{x}_{t+1} \rangle + \langle \eta_t \mathbf{g}_t, \mathbf{x}_t - \mathbf{x}_{t+1} \rangle \\
&= B_\psi(\mathbf{u}; \mathbf{x}_t) - B_\psi(\mathbf{u}; \mathbf{x}_{t+1}) - B_\psi(\mathbf{x}_{t+1}; \mathbf{x}_t) + \langle \eta_t \mathbf{g}_t, \mathbf{x}_t - \mathbf{x}_{t+1} \rangle \\
&\leq B_\psi(\mathbf{u}; \mathbf{x}_t) - B_\psi(\mathbf{u}; \mathbf{x}_{t+1}) - \frac{\lambda}{2}\|\mathbf{x}_t - \mathbf{x}_{t+1}\|^2 + \eta_t \|\mathbf{g}_t\|_* \|\mathbf{x}_t - \mathbf{x}_{t+1}\| \\
&\leq B_\psi(\mathbf{u}; \mathbf{x}_t) - B_\psi(\mathbf{u}; \mathbf{x}_{t+1}) + \frac{\eta_t^2}{2\lambda}\|\mathbf{g}_t\|_*^2.
\end{aligned}$$

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\begin{aligned}
\langle \eta_t \mathbf{g}_t, \mathbf{x}_t - \mathbf{u} \rangle &= -\langle \eta_t \mathbf{g}_t + \nabla\psi(\mathbf{x}_{t+1}) - \nabla\psi(\mathbf{x}_t), \mathbf{u} - \mathbf{x}_{t+1} \rangle \\
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&\leq \langle \nabla\psi(\mathbf{x}_{t+1}) - \nabla\psi(\mathbf{x}_t), \mathbf{u} - \mathbf{x}_{t+1} \rangle + \langle \eta_t \mathbf{g}_t, \mathbf{x}_t - \mathbf{x}_{t+1} \rangle \\
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\langle \eta_t \mathbf{g}_t, \mathbf{x}_t - \mathbf{u} \rangle &= -\langle \eta_t \mathbf{g}_t + \nabla\psi(\mathbf{x}_{t+1}) - \nabla\psi(\mathbf{x}_t), \mathbf{u} - \mathbf{x}_{t+1} \rangle \\
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&\le B_\psi(\mathbf{u}; \mathbf{x}_t) - B_\psi(\mathbf{u}; \mathbf{x}_{t+1}) + \frac{\eta_t^2}{2\lambda} \|\mathbf{g}_t\|_*^2.
\end{aligned}$$

Hint

$$ax - \frac{b}{2}x^2 \leq \frac{a^2}{2b}, \text{ for } x \in \mathbb{R} \text{ and } a, b > 0.$$

Main Theorem

Main Theorem I

- Set $\mathbf{x}_1 \in V$ such that ψ is differentiable in $\mathbf{x}_1$.
- Assume that $\eta_{t+1} \leq \eta_t$ for $t = 1, \ldots, T$.

Then, under the assumption in the Main Lemma and $\forall \mathbf{u} \in V$, we have

$$\sum_{t=1}^{T} (f_t(\mathbf{x}_t) - f_t(\mathbf{u})) \leq \max_{1 \leq t \leq T} \frac{B_\psi(\mathbf{u}; \mathbf{x}_t)}{\eta_T} + \frac{1}{2\lambda} \sum_{t=1}^{T} \eta_t \|\mathbf{g}_t\|_*^2.$$

Proof of Main Theorem I

$$\sum_{t=1}^{T}(f_t(\mathbf{x}_t) - f_t(\mathbf{u})) \le \sum_{t=1}^{T}\left(\frac{1}{\eta_t}B_\psi(\mathbf{u};\mathbf{x}_t) - \frac{1}{\eta_t}B_\psi(\mathbf{u};\mathbf{x}_{t+1})\right) + \sum_{t=1}^{T}\frac{\eta_t^2}{2\lambda}\|\mathbf{g}_t\|_*^2$$

$$= \frac{1}{\eta_1}B_\psi(\mathbf{u};\mathbf{x}_1) - \frac{1}{\eta_T}B_\psi(\mathbf{u};\mathbf{x}_{T+1}) + \sum_{t=1}^{T-1}\left(\frac{1}{\eta_{t+1}} - \frac{1}{\eta_t}\right)B_\psi(\mathbf{u};\mathbf{x}_{t+1}) + \sum_{t=1}^{T}\frac{\eta_t}{2\lambda}\|\mathbf{g}_t\|_*^2$$

$$\le \frac{1}{\eta_1}D^2 + D^2\sum_{t=1}^{T-1}\left(\frac{1}{\eta_{t+1}} - \frac{1}{\eta_t}\right) + \sum_{t=1}^{T}\frac{\eta_t}{2\lambda}\|\mathbf{g}_t\|_*^2$$

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$$= \frac{D^2}{\eta_T} + \sum_{t=1}^{T}\frac{\eta_t}{2\lambda}\|\mathbf{g}_t\|_*^2,$$

where $D^2 := \max_{1\le t\le T} B_\psi(\mathbf{u};\mathbf{x}_t)$.

What we can learn from OMD?

- OMD allows us to prove regret guarantees depending on arbitrary norms $\|\cdot\|$ and $\|\cdot\|_*$.
- The primal norm: measure in the feasible space.
- The dual norm: measuring the gradients.

Using a Fixed Learning Rate $\eta_t = \eta$

- Assume that f_t is L-Lipschitz continuous $\Rightarrow \|\mathbf{g}_t\|_*^2 = \|\mathbf{g}_t\|_2^2 \leq L^2$.
- To minimize $\frac{D^2}{\eta_T} + \sum_{t=1}^{T} \frac{\eta_t}{2\lambda}\|\mathbf{g}_t\|_*^2 = \frac{D^2}{\eta} + \frac{T\eta L^2}{2\lambda}$.
    - Take the derivative w.r.t. η and get root: $\Rightarrow \frac{D^2}{\eta^2} = \frac{TL^2}{2\lambda}$, $\eta = \frac{\sqrt{2\lambda}D}{L\sqrt{T}}$
    - Then the regret is $\frac{DL\sqrt{2T}}{\sqrt{\lambda}}$.

Using Adaptive Learning Rate

- Set $\eta_t = \frac{D\sqrt{\lambda}}{\sqrt{\sum_{i=1}^{t} \|\mathbf{g}_i\|_2^2}}$.
- We can show that

$$\sum_{t=1}^{T} \frac{\eta_t}{2\lambda} \|\mathbf{g}_t\|_2^2 = \frac{D}{2\sqrt{\lambda}} \sum_{t=1}^{T} \frac{\|\mathbf{g}_t\|_2^2}{\sum_{i=1}^{t} \|\mathbf{g}_i\|_2^2}$$

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Remark on Main Theorem I

- The regret bound depends on arbitrary couple of dual norms $\|\cdot\|$ and $\|\cdot\|_*$.
    - Usually, the primal norm is used to measure the feasible set V or the distance between the competitor and the initial point.
    - The dual norm will be used to measure the gradients.

Outline

Duality Strong Convexity/Smoothness

Theorem

Let

- $f : \mathbb{R}^d \mapsto (-\infty, +\infty]$ be a closed and convex function
- $\text{dom}(\partial f) \neq \emptyset$

Then for $\lambda > 0$, f is λ-strongly convex w.r.t. $\|\cdot\|$ iff f^* is $\frac{1}{\lambda}$-smooth w.r.t. $\|\cdot\|_*$ on $\mathbb{R}^d$.

- f^* is differentiable.

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- f^* is differentiable.
  - f is proper, closed and **strongly** convex $\Rightarrow$ the maximizer $\mathbf{x}^*$ of $\max_{\mathbf{x}} \langle \boldsymbol{\theta}, \mathbf{x} \rangle - f(\mathbf{x})$ exists and is **unique** (p. 24).

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  - Hence, $\mathbf{x}^* \in \partial f^*(\boldsymbol{\theta})$.

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  - Hence, $\mathbf{x}^* \in \partial f^*(\boldsymbol{\theta})$.
  - Assume another $\mathbf{x}' \in \partial f^*(\boldsymbol{\theta}) \Rightarrow f^*(\boldsymbol{\theta}) = \langle \boldsymbol{\theta}, \mathbf{x}' \rangle - f(\mathbf{x}')$.
  - By the uniqueness of the maximizer, we have $\mathbf{x}^* = \mathbf{x}'$.

($\Rightarrow$ contd.)

- For any $\boldsymbol{\theta}_1, \boldsymbol{\theta}_2$, let $\mathbf{x}_1 = \nabla f^*(\boldsymbol{\theta}_1), \mathbf{x}_2 = \nabla f^*(\boldsymbol{\theta}_2)$.
    - Then we have $\boldsymbol{\theta}_1 \in \partial f(\mathbf{x}_1)$, $\boldsymbol{\theta}_2 \in \partial f(\mathbf{x}_2)$.
- By the strong convexity, we have

$$
\begin{aligned}
f(\mathbf{x}_2) &\geq f(\mathbf{x}_1) + \langle \boldsymbol{\theta}_1, \mathbf{x}_2 - \mathbf{x}_1 \rangle + \frac{\lambda}{2}\|\mathbf{x}_1 - \mathbf{x}_2\|^2 \\
f(\mathbf{x}_1) &\geq f(\mathbf{x}_2) + \langle \boldsymbol{\theta}_2, \mathbf{x}_1 - \mathbf{x}_2 \rangle + \frac{\lambda}{2}\|\mathbf{x}_1 - \mathbf{x}_2\|^2
\end{aligned}
$$

$$
\Rightarrow \quad \langle \boldsymbol{\theta}_2 - \boldsymbol{\theta}_1, \mathbf{x}_1 - \mathbf{x}_2 \rangle \geq \lambda \|\mathbf{x}_1 - \mathbf{x}_2\|^2.
$$

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$$\Rightarrow \|\boldsymbol{\theta}_1 - \boldsymbol{\theta}_2\|_* \|\mathbf{x}_1 - \mathbf{x}_2\| \geq \langle \boldsymbol{\theta}_2 - \boldsymbol{\theta}_1, \mathbf{x}_1 - \mathbf{x}_2 \rangle \geq \lambda \|\mathbf{x}_1 - \mathbf{x}_2\|^2.$$

($\Leftarrow$)

- Assume that f^* is $\frac{1}{\lambda}$-smooth w.r.t. $\|\cdot\|_*$ on $\mathbb{R}^d$.
- Let $\mathbf{y} \in \text{dom}(\partial f)$ and $\mathbf{u} \in \partial f(\mathbf{y})$.
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- Define $\phi(\boldsymbol{\theta}) := f^*(\boldsymbol{\theta} + \mathbf{u}) - f^*(\mathbf{u}) - \langle \boldsymbol{\theta}, \nabla f^*(\mathbf{u}) \rangle$.

Recall that if $f : V \mapsto \mathbb{R}$ is M-smooth, then for any $\mathbf{x}, \mathbf{y} \in V$ we have

$$|f(\mathbf{y}) - f(\mathbf{x}) - \langle \nabla f(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle| \leq \frac{M}{2} \|\mathbf{y} - \mathbf{x}\|^2.$$

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- Hence, $\phi(\boldsymbol{\theta}) \leq \frac{1}{2\lambda} \|\boldsymbol{\theta}\|_*^2 := \hat{\phi}(\theta)$.

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$$\phi^*(\mathbf{x}) \ \ge\ \hat{\phi}^*(\mathbf{x}) \ =\ \sup_{\boldsymbol{\theta}} \langle \boldsymbol{\theta}, \mathbf{x} \rangle - \frac{1}{2\lambda} \|\boldsymbol{\theta}\|_*^2$$

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$$\begin{aligned}
\phi^*(\mathbf{x}) \geq \hat{\phi}^*(\mathbf{x}) &= \sup_{\boldsymbol{\theta}} \langle \boldsymbol{\theta}, \mathbf{x} \rangle - \frac{1}{2\lambda}\|\boldsymbol{\theta}\|_*^2 \leq \sup_{\boldsymbol{\theta}} \|\boldsymbol{\theta}\|_* \|\mathbf{x}\| - \frac{1}{2\lambda}\|\boldsymbol{\theta}\|_*^2 \\
&= \sup_{\boldsymbol{\theta}} \left(-\frac{1}{2\lambda}(\|\boldsymbol{\theta}\|_*^2 - 2\lambda\|\mathbf{x}\|\|\boldsymbol{\theta}\|_* + (\lambda\|\mathbf{x}\|)^2) \right) \\
&= \frac{\lambda}{2}\|\mathbf{x}\|^2.
\end{aligned}$$

($\Leftarrow$) Recall that $\phi(\boldsymbol{\theta}) := f^*(\boldsymbol{\theta} + \mathbf{u}) - f^*(\mathbf{u}) - \langle \boldsymbol{\theta}, \nabla f^*(\mathbf{u}) \rangle$.

- Calculate $\phi^*(\mathbf{x})$: (Let $\mathbf{v} = \boldsymbol{\theta} + \mathbf{u}$)

$$
\begin{aligned}
\phi^*(\mathbf{x}) &= \sup_{\boldsymbol{\theta}} \left(\langle \boldsymbol{\theta}, \mathbf{x} \rangle - f^*(\boldsymbol{\theta} + \mathbf{u}) + f^*(\mathbf{u}) + \langle \boldsymbol{\theta}, \nabla f^*(\mathbf{u}) \rangle \right) \\
&= f^*(\mathbf{u}) - \langle \mathbf{u}, \mathbf{x} + \nabla f^*(\mathbf{u}) \rangle + \sup_{\mathbf{v}} \left(\langle \mathbf{v}, \mathbf{x} + \nabla f^*(\mathbf{u}) \rangle - f^*(\mathbf{v}) \right) \\
&= f^*(\mathbf{u}) - \langle \mathbf{u}, \mathbf{x} + \nabla f^*(\mathbf{u}) \rangle + f(\mathbf{x} + \nabla f^*(\mathbf{u})) \\
&= f^*(\mathbf{u}) - \langle \mathbf{u}, \mathbf{x} \rangle - \langle \mathbf{u}, \nabla f^*(\mathbf{u}) \rangle + f(\mathbf{x} + \nabla f^*(\mathbf{u})) \\
&= -\langle \mathbf{u}, \mathbf{x} \rangle - f(\nabla f^*(\mathbf{u})) + f(\mathbf{x} + \nabla f^*(\mathbf{u})) \\
&= f(\mathbf{x} + \mathbf{y}) - f(\mathbf{y}) - \langle \mathbf{u}, \mathbf{x} \rangle
\end{aligned}
$$

- Using $\phi^*(\mathbf{x}) \geq \frac{\lambda}{2} \|\mathbf{x}\|^2$ then we are done.

First-order optimality

Theorem

For a function $f : \mathbb{R}^d \mapsto (-\infty, +\infty]$, we have

$$\mathbf{x}^* \in \arg\min_{\mathbf{x}\in\mathbb{R}^d} f(\mathbf{x}) \text{ if and only if } \mathbf{0} \in \partial f(\mathbf{x}^*).$$

$$\begin{aligned}\mathbf{x}^* \in \arg\min_{\mathbf{x}\in\mathbb{R}^d} f(\mathbf{x}) \quad &\Leftrightarrow \quad \forall \mathbf{y} \in \mathbf{R}^d, f(\mathbf{y}) \geq f(\mathbf{x}^*) + \langle \mathbf{0}, \mathbf{y} - \mathbf{x}^* \rangle \\ &\Leftrightarrow \quad \mathbf{0} \in \partial f(\mathbf{x}^*).\end{aligned}$$

The OMD update in terms of duality mappings

Theorem (OMD & Duality Mappings)

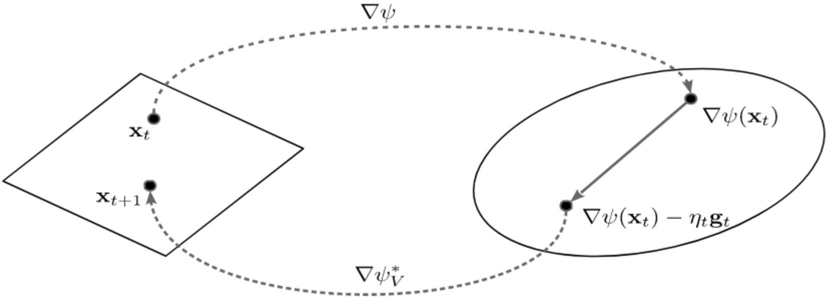
- Let B_ψ be the Bregman divergence w.r.t. $\psi : X \mapsto \mathbb{R}$, where ψ is closed and λ-strongly convex for $\lambda > 0$.
- Define $\mathbf{x}_{t+1} := \arg\min_{\mathbf{x} \in V} \langle \mathbf{g}_t, \mathbf{x} \rangle + \frac{1}{\eta_t} B_\psi(\mathbf{x}; \mathbf{x}_t)$, and assume that ψ is differentiable in $\mathbf{x}_t$ and $\mathbf{x}_{t+1}$.

Then, for any $\mathbf{g}_t \in \mathbb{R}^d$, we have

$$\mathbf{x}_{t+1} = \nabla \psi_V^*(\nabla \psi(\mathbf{x}_t) - \eta_t \mathbf{g}_t),$$

where $\psi_V := \psi + i_V$ which restricts ψ to V.

The OMD update in terms of duality mappings



$$\begin{aligned}\mathbf{x}_{t+1} &:= \arg\min_{\mathbf{x}\in V} \langle \mathbf{g}_t, \mathbf{x}\rangle + \frac{1}{\eta_t} B_\psi(\mathbf{x}; \mathbf{x}_t) \\ &= \mathbf{x}_{t+1} = \nabla\psi_V^*(\nabla\psi(\mathbf{x}_t) - \eta_t \mathbf{g}_t).\end{aligned}$$

Proof of the main theorem (1/2)

$$\begin{aligned}
\mathbf{x}_{t+1} &:= \arg\min_{\mathbf{x}\in V} \langle \mathbf{g}_t, \mathbf{x}\rangle + \frac{1}{\eta_t} B_\psi(\mathbf{x}; \mathbf{x}_t) \\
&= \arg\min_{\mathbf{x}\in V} \eta_t \langle \mathbf{g}_t, \mathbf{x}\rangle + B_\psi(\mathbf{x}; \mathbf{x}_t) \\
&= \arg\min_{\mathbf{x}\in V} \eta_t \langle \mathbf{g}_t, \mathbf{x}\rangle + \psi(\mathbf{x}) - \psi(\mathbf{x}_t) - \langle \nabla\psi(\mathbf{x}_t), \mathbf{x} - \mathbf{x}_t\rangle \\
&= \arg\min_{\mathbf{x}\in V} \langle \eta_t \mathbf{g}_t - \nabla\psi(\mathbf{x}_t), \mathbf{x}\rangle + \psi(\mathbf{x}).
\end{aligned}$$

By the first-order optimality condition, we have

$$\begin{aligned}
\mathbf{0} &\in \eta_t \mathbf{g}_t + \nabla\psi(\mathbf{x}_{t+1}) - \nabla\psi(\mathbf{x}_t) + \partial i_V(\mathbf{x}_{t+1}) \\
\nabla\psi(\mathbf{x}_t) - \eta_t \mathbf{g}_t &\in (\nabla\psi + \partial i_V)(\mathbf{x}_{t+1}) \subseteq \partial\psi_V(\mathbf{x}_{t+1})
\end{aligned}$$

Proof of the main theorem (1/2)

$$
\begin{aligned}
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&= \arg\min_{\mathbf{x}\in V} \eta_t \langle \mathbf{g}_t, \mathbf{x}\rangle + \psi(\mathbf{x}) - \psi(\mathbf{x}_t) - \langle \nabla\psi(\mathbf{x}_t), \mathbf{x} - \mathbf{x}_t\rangle \\
&= \arg\min_{\mathbf{x}\in V} \langle \eta_t \mathbf{g}_t - \nabla\psi(\mathbf{x}_t), \mathbf{x}\rangle + \psi(\mathbf{x}).
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$$

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$$
\begin{aligned}
\mathbf{0} &\in \eta_t \mathbf{g}_t + \nabla\psi(\mathbf{x}_{t+1}) - \nabla\psi(\mathbf{x}_t) + \partial i_V(\mathbf{x}_{t+1}) \\
\nabla\psi(\mathbf{x}_t) - \eta_t \mathbf{g}_t &\in (\nabla\psi + \partial i_V)(\mathbf{x}_{t+1}) \subseteq \partial\psi_V(\mathbf{x}_{t+1})
\end{aligned}
$$

Hence, $\mathbf{x}_{t+1} \in \partial\psi_V^*(\nabla\psi(\mathbf{x}_t) - \eta_t \mathbf{g}_t)$.

Proof of the main theorem (2/2)

- Note that $\psi_V := \psi + i_V$ is proper, λ-strongly convex and closed.
  - $\partial\psi_V^* = \{\nabla\psi_V^*\}$.
- Therefore, since $\mathbf{x}_{t+1} \in \partial\psi_V^*(\nabla\psi(\mathbf{x}_t) - \eta_t\mathbf{g}_t)$, we have $\mathbf{x}_{t+1} = \nabla\psi_V^*(\nabla\psi(\mathbf{x}_t) - \eta_t\mathbf{g}_t)$.

Example (1/2)

- $\psi : \mathbb{R}^d \mapsto \mathbb{R}$, $\psi(\mathbf{x}) = \frac{1}{2}\|\mathbf{x}\|_2^2$
- $V = \{\mathbf{x} \in \mathbb{R}^d : \|\mathbf{x}\|_2 \leq 1\}$.
- $\psi_V := \psi + i_V$.

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- $\psi_V^*(\boldsymbol{\theta}) = \sup_{\mathbf{x} \in V} \langle \boldsymbol{\theta}, \mathbf{x} \rangle - \frac{1}{2}\|\mathbf{x}\|_2^2$.
- Assume $\boldsymbol{\theta} \neq \mathbf{0}$ (otherwise, trivially $\psi_V^*(\boldsymbol{\theta}) = \mathbf{0}$).

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- $\psi_V^*(\boldsymbol{\theta}) = \sup_{\mathbf{x} \in V} \langle \boldsymbol{\theta}, \mathbf{x} \rangle - \frac{1}{2}\|\mathbf{x}\|_2^2$.
- Assume $\boldsymbol{\theta} \neq \mathbf{0}$ (otherwise, trivially $\psi_V^*(\boldsymbol{\theta}) = \mathbf{0}$).
- For any $\mathbf{x} \in V$, there exists $\mathbf{q}$ and α such that $\mathbf{x} = \alpha \frac{\boldsymbol{\theta}}{\|\boldsymbol{\theta}\|_2} + \mathbf{q}$ and $\langle \mathbf{q}, \boldsymbol{\theta} \rangle = 0$.
    - $\nabla \psi(\mathbf{x}) = \mathbf{x}$.

Example (2/2)

$$\begin{aligned}\sup_{\mathbf{x}\in V}\langle\boldsymbol{\theta},\mathbf{x}\rangle-\frac{1}{2}\|\mathbf{x}\|_2^2 &= \sup_{\alpha,\mathbf{q}:\alpha\frac{\boldsymbol{\theta}}{\|\boldsymbol{\theta}\|_2}+\mathbf{q}\in V,\langle\mathbf{q},\boldsymbol{\theta}\rangle=0}\alpha\|\boldsymbol{\theta}\|_2-\frac{\alpha^2}{2}-\frac{1}{2}\|\mathbf{q}\|_2^2\\ &= \sup_{-1\le\alpha\le1}\alpha\|\boldsymbol{\theta}\|_2-\frac{\alpha^2}{2}\\ &= \sup_{-1\le\alpha\le1}-\frac{1}{2}(\alpha-\|\boldsymbol{\theta}\|_2)^2+\frac{1}{2}\|\boldsymbol{\theta}\|_2^2.\end{aligned}$$

- Solving the constrained optimization problem, we have $\alpha^*=\min(1,\|\boldsymbol{\theta}\|_2)$.
- Hence,

$$\psi_V^*(\boldsymbol{\theta})=\begin{cases}\frac{1}{2}\|\boldsymbol{\theta}\|_2^2, & \text{if } \|\boldsymbol{\theta}\|_2\le1\\ \|\boldsymbol{\theta}\|_2-\frac{1}{2}, & \text{if } \|\boldsymbol{\theta}\|_2>1\end{cases},$$

Example (2/2)

$$\begin{aligned}
\sup_{\mathbf{x}\in V}\langle\boldsymbol{\theta},\mathbf{x}\rangle-\frac{1}{2}\|\mathbf{x}\|_2^2 &= \sup_{\alpha,\mathbf{q}:\alpha\frac{\boldsymbol{\theta}}{\|\boldsymbol{\theta}\|_2}+\mathbf{q}\in V,\langle\mathbf{q},\boldsymbol{\theta}\rangle=0} \alpha\|\boldsymbol{\theta}\|_2-\frac{\alpha^2}{2}-\frac{1}{2}\|\mathbf{q}\|_2^2 \\
&= \sup_{-1\le\alpha\le1} \alpha\|\boldsymbol{\theta}\|_2-\frac{\alpha^2}{2} \\
&= \sup_{-1\le\alpha\le1} -\frac{1}{2}(\alpha-\|\boldsymbol{\theta}\|_2)^2+\frac{1}{2}\|\boldsymbol{\theta}\|_2^2.
\end{aligned}$$

- Solving the constrained optimization problem, we have $\alpha^*=\min(1,\|\boldsymbol{\theta}\|_2)$.
- Hence,

$$\psi_V^*(\boldsymbol{\theta})=\begin{cases}\frac{1}{2}\|\boldsymbol{\theta}\|_2^2, & \text{if } \|\boldsymbol{\theta}\|_2\le 1\\ \|\boldsymbol{\theta}\|_2-\frac{1}{2}, & \text{if } \|\boldsymbol{\theta}\|_2>1\end{cases},$$

$$\nabla\psi_V^*(\boldsymbol{\theta})=\begin{cases}\boldsymbol{\theta}, & \text{if } \|\boldsymbol{\theta}\|_2\le 1\\ \frac{\boldsymbol{\theta}}{\|\boldsymbol{\theta}\|_2}, & \text{if } \|\boldsymbol{\theta}\|_2>1\end{cases}=\Pi_V(\boldsymbol{\theta}).$$

Remark

- OMD extends the OSD to non-Euclidean norms.
- The dual norm is used to measure a gradient.
- Gradients live in the dual space, different from the predictions $\mathbf{x}_t$.
- In the OSD, the dual space coincides with the primal space.
- The ways we go from one space to the other: $\nabla\psi$ and $\nabla\psi_V^*$.

Discussions

Test

We have the following formula:

$$J_{\mathcal{G}}(\mathbf{z}_u) = -\log(\sigma(\mathbf{z}_u^\top \mathbf{z}_v)) - Q \cdot \mathbb{E}_{v_n \sim P_n(v)} \log(\sigma(-\mathbf{z}_u^\top \mathbf{z}_{v_n}))$$