Randomized Algorithms (2026 Spring)

Assignment Set 2*

Chuang-Chieh Lin

Department of Computer Science and Engineering,
National Taiwan Ocean University

1. Let's say we have a set S of $n = 2k + 1$ integers for some positive integer k.
   (a) If we pick one number x from S uniformly at random, what is the probability that x is the median of S?
   (b) If we pick three numbers x, y, z from S uniformly at random (with replacement), what is the probability that the median of $\{x, y, z\}$ is the median of S?

2. Suppose that S is a set of $k > 0$ integers such that $k = o(n)$. Please show by providing a counterexample that sorting k by an optimal comparison-based algorithm is not necessarily in $o(n)$ time.

3. Let X be a number chosen uniformly at random from $[1, n]$. Find $\mathrm{Var}[X]$.

4. Suppose that we roll a standard fair die 100 times. Let X be the sum of the numbers that appear over the 100 rolls. Use Chebyshev's inequality to bound

$$\Pr\big[|X - 350| \geq 50\big].$$

5. A simple model of the stock market suggests that, each day, a stock with price q will increase by a factor $r > 1$ to qr with probability p and will fall to q/r with probability $1 - p$. Assuming we start with a stock with price 1, find a formula for the **expected value** and the **variance** of the price of the stock after d days.

6. Let X be a geometric random variable with parameter p. Please compute its second moment $\mathbb{E}[X^2]$.

7. Suppose that S is a set of $n > 1$ distinct numbers. Let z be a number picked uniformly at random from S and define $S_1 := \{x \in S : x < z\}$, $S_2 := \{x \in S : x > z\}$. Please show that

$$\mathbb{E}[\max\{|S_1|, |S_2|\}] \leq \frac{3}{4}n.$$

8. Let Y be a nonnegative integer-valued random variable with positive expectation. Prove that

$$\frac{\mathbb{E}[Y]^2}{\mathbb{E}[Y^2]} \leq \Pr[Y \neq 0] \leq \mathbb{E}[Y].$$

9. Prove that $\mathbb{E}[X^k] \geq \mathbb{E}[X]^k$ for any even integer $k > 1$.

* We will select five of them for the Quiz.