Randomized Algorithms (2026 Spring)

Assignment Set 3*

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1. Alice and Bob play checkers often. Alice is a better player, so the probability that she wins any given game is 0.6, independent of all other games. They decide to play a tournament of n games. Bound the probability that Alice loses the tournament using a Chernoff bound.

2. In an election with two candidates using paper ballots, each vote is independently misrecorded with probability $p = 0.02$. Use a Chernoff bound to give an upper bound on the probability that more than 4% of the votes are misrecorded in an election of 1,000,000 ballots.

3. Consider the problem of throwing m balls into n bins uniformly at random. Let $X_i^{(m)}$ be random variable denoting the number of balls in the ith bin, for $i = 1, 2, \ldots, n$, Let $Y_1^{(m)}, Y_2^{(m)}, \ldots, Y_n^{(m)}$ be independent Poisson random variables with mean m/n. Let $f(x_1, x_2, \ldots, x_n)$ be a nonnegative function such that $\mathbb{E}[f(X_1^{(m)}, X_2^{(m)}, \ldots, X_n^{(m)})]$ is monotonically increasing in m.
   (a) Let $s_t := \mathbb{E}[f(X_1^{(t)}, X_2^{(t)}, \ldots, X_n^{(t)})]$ and $Z := \sum_{i=1}^n Y_i^{(m)}$. Please show that
   $$\mathbb{E}[f(Y_1^{(m)}, Y_2^{(m)}, \ldots, Y_n^{(m)})] = \sum_{t=0}^{\infty} s_t \Pr[Z = t].$$
   (b) Given that s_t is monotonically increasing in t, please show that
   $$\mathbb{E}[f(Y_1^{(m)}, Y_2^{(m)}, \ldots, Y_n^{(m)})] \geq s_m \cdot \Pr[Z \geq m].$$
   (c) Assume that $m \geq 1$ is an integer and $Z \sim \text{Poisson}(m)$. Please prove that
   $$\Pr[Z \geq m] \geq \frac{1}{2}.$$

4. Use the Taylor expansion
   $$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$
   to prove that, for any x with $|x| \leq 1$,
   $$e^x(1 - x^2) \leq 1 + x \leq e^x.$$

5. Suppose that n balls are thrown independently and uniformly at random into n bins.

* We will select three of them for the Quiz.

(a) Find the conditional probability that bin 1 has one ball given that exactly one ball fell into the first three bins.

(b) Find the conditional expectation of the number of balls in bin 1 under the condition that bin 2 received no balls.