Randomized Algorithms (2026 Spring)
Assignment Set 4*

Chuang-Chieh Lin

Department of Computer Science and Engineering,
National Taiwan Ocean University

1. Let $N = \lfloor (n \ln n + cn)/2 \rfloor$ for some constant $c > 0$. Prove that the expected number of isolated vertices in a graph sampled in $G_{n,N}$ uniformly at random converges to e^{-c} as $n \to \infty$.

2. Consider the two-state Markov chain with the following transition matrix

$$\mathbf{P} = \begin{bmatrix} p & 1-p \\ 1-p & p \end{bmatrix}.$$

   Find a simple expression for $P^t_{0,0}$.

3. In studying the 2-SAT algorithm, we considered a one-dimensional random walk with a completely reflecting boundary at 0. That is, whenever position 0 is reached, with probability 1 the walk moves to position 1 at the next step. Consider now a random walk with a partially reflecting boundary at 0. Whenever position 0 is reached, with probability 1/2 the walk moves to position 1, and with probability 1/2 the walk stays at 0. Everywhere else the random walk moves either up or down by 1, each with probability 1/2. Find the **expected number of moves to reach n, starting from position i**, and using a random walk with a partially reflecting boundary.

4. Let X_n be the sum of n independent rolls of a fair die. Show that, for any $k \geq 2$,

$$\lim_{n \to \infty} \Pr[X_n \text{ is divisible by } k] = \frac{1}{k}.$$

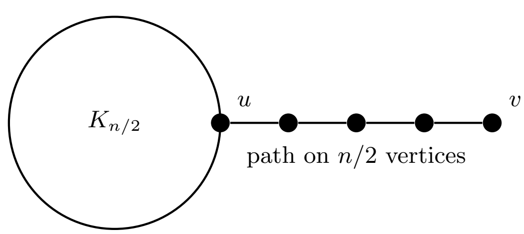

Figure 1: Lollipop graph.

5. The *lollipop graph* on n vertices is a clique on $n/2$ vertices connected to a path on $n/2$ vertices, as shown in Figure 1. The node u is a part of both the clique and the path. Let v denote the other end of the path.

* We will select three or four of them for the Quiz.

(a) Show that the expected covering time of a random walk starting at v is $O(n^2)$.

(b) Show that the expected covering time for a random walk starting at u is $O(n^3)$.

6. (a) Prove that, for every integer n, there exists a coloring of the edges of the complete graph K_n by two colors so that the total number of monochromatic copies of K_4 is at most

$$\binom{n}{4} 2^{-5}.$$

(b) Give a randomized algorithm for finding a coloring with at most

$$\binom{n}{4} 2^{-5}$$

monochromatic copies of K_4 that runs in expected time polynomial in n.

(c) Show how to construct such a coloring deterministically in polynomial time using the method of conditional expectations.