Randomized Algorithms (2026 Spring)

Final Exam

1. (25%) Please **analyze the time complexity of Algorithm Revised-Rand-Median** (Fig. 1) and, following the style of the printed lecture notes, prove an upper bound on the probability of outputting FAIL. Moreover, please **give an upper bound (as tight as possible) on the probability of outputting Fail.**

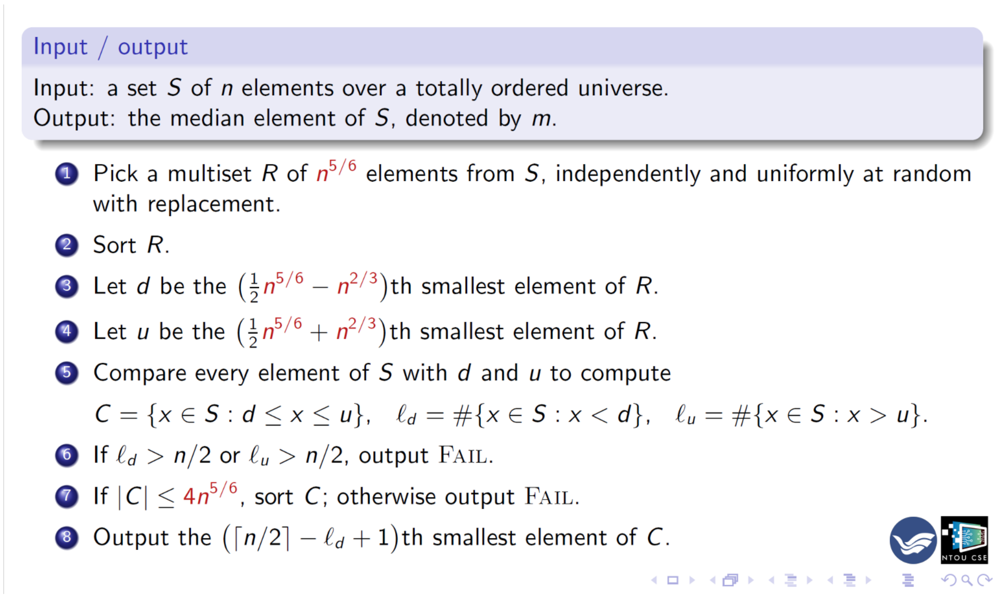

Figure 1: Algorithm Revised-Rand-Median.

2. (10%) Alice and Bob play checkers often. Alice is a better player, so the probability that she wins any given game is 0.6, independent of all other games. They decide to play a tournament of n games. Bound the probability that Alice wins fewer than $n/2$ games using a Chernoff bound.

3. (10%) Let $N = \lfloor (n \ln n + cn)/2 \rfloor$ for some constant $c > 0$. Prove that the expected number of isolated vertices in a graph sampled in $G_{n,N}$ uniformly at random converges to e^{-c} as $n \to \infty$.

4. (10%) We have known that for collecting n types of coupons, each of which appears in a draw with equal probability, the expected number of draws for collecting all types is at most $cn \log n$ for some constant $c > 1/2$. Suppose that we make $R = 2cn \ln n$ independent draws. By the Markov inequality, the probability of failing to collect all types is at most $1/2$. Please provide a better (i.e., asymptotically smaller) upper bound on this probability of failure. (*Hint:* Use a union-bound argument to show that the failure probability is at most n^{1-2c}.)

5. (15%) To sort a list of n distinct numbers, randomized QuickSort is efficient when the pivot is not too far from the median. The rank of the smallest element is 1. For simplicity, assume n is divisible by 3.
   (a) (5%) Suppose the pivot is chosen uniformly at random from the n numbers. Give an upper bound on the probability that the pivot has rank smaller than $n/3$ or larger than $2n/3$.
   (b) (10%) Suppose instead that we independently choose three numbers uniformly at random from the list, and use the median of the three chosen numbers as the pivot. Give an upper bound on the probability that this pivot has rank smaller than $n/3$ or larger than $2n/3$.

6. (10%) Suppose that we have a biased coin that shows HEAD with probability p, where $0 < p < 1$, and assume all tosses are independent. Devise an algorithm using this biased coin that outputs HEAD with probability $1/2$, i.e., simulates a fair coin. Prove that your algorithm is correct.

7. (10%) Consider the Lovász Local Lemma in the reference lecture. Please prove that if no variable in a k-SAT formula appears in more than $T = 2^k/4k$ clauses, then the formula has a satisfying assignment. (Note: Assume each clause contains exactly k distinct variables, and each variable appears in at most $\lfloor T \rfloor$ clauses, counting both positive and negative occurrences.)

8. (15%) Consider the randomized local-search algorithm for satisfiable 2-SAT formulas. The algorithm starts from an arbitrary truth assignment. If the current assignment does not satisfy all clauses, it chooses an arbitrary unsatisfied clause, chooses one of its two literals uniformly at random, and flips the corresponding variable. Assume that the 2-SAT formula has n variables and is satisfiable. Fix one satisfying assignment S. Let A_i be the current assignment after i repair steps, and define $X_i = \#\{x_\ell : A_i(x_\ell) = S(x_\ell)\}$. Suppose we have known that whenever $0 < X_i < n$, the probability that one repair step increases X_i by 1 is at least $1/2$.
   (a) (5%) Consider the pessimistic Markov chain (Y_i) on $\{0, 1, \ldots, n\}$ defined by $\Pr[Y_{i+1} = 1 \mid Y_i = 0] = 1$, and for $1 \le j \le n-1$, $\Pr[Y_{i+1} = j+1 \mid Y_i = j] = \frac{1}{2}$, $\Pr[Y_{i+1} = j-1 \mid Y_i = j] = \frac{1}{2}$. Let h_j be the expected number of steps for this chain to reach n starting from $Y_0 = j$. Please Derive the system of equations satisfied by $h_0, h_1, \ldots, h_n$.
   (b) (10%) Solve the system and prove that $h_j = n^2 - j^2$ which concludes that the expected number of repair steps of the randomized 2-SAT algorithm is at most n^2.

9. (10%) Consider m people in a room and n possible birthdays ($m \le n$). Assume each person's birthday is chosen independently and uniformly from the n days. We regard people as balls and days as bins. Show that the probability that no two people share the same birthday is

$$\prod_{j=1}^{m-1} \left(1 - \frac{j}{n}\right).$$

   Then show that this probability is approximately $e^{-m^2/(2n)}$.