Randomized Algorithms

Introduction

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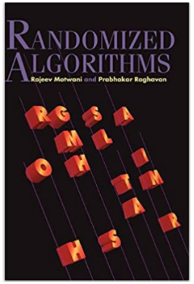
Self Introduction

- **Ph.D.**: CSIE, National Chung Cheng University, 2011.
  - DAAD-NSC Sandwich Program (2007−2008).
  - Dissertation supervisors: Maw-Shang Chang & Peter Rossmanith (RWTH Aachen)

- **Postdoc** in Genomics Research Center, Academia Sinica (2011−2014).
- **Postdoc** in Institute of Information Science, Academia Sinica (2014−2018).
- **Quantitative Analyst (intern)** of Point72/Cubist Systematic Strategies (2018−2019).
- **Quantitative Analyst** of Seth Technologies Inc. (2020−2021/01).

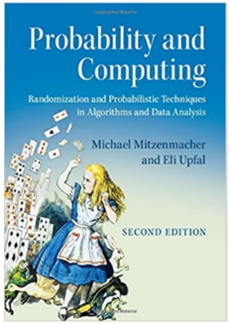
Textbooks and Materials

- Textbooks:
  - *Randomized Algorithms*. Motwani, R. and Raghavan, P., 1995. Cambridge University Press.



  - Probability and Computing: Randomization and Probabilistic Techniques in Algorithms and Data Analysis. Second Edition. M. Mitzenmacher and E. Upfal, Cambridge University Press, 2017.

- Other materials:
  - Prepared slides.



Prerequisites

- Basic undergraduate courses in
  - Algorithms
  - Data structures
  - Probability theory
  - Discrete mathematics
  - Linear Algebra
- Motivation.
- Curiosity.

Topics

- Introduction & the Min-Cut Problem
- Examples of Probability Paradoxes
- Probability Prerequisites
- Coupon Collector's Problem
- The Secretary Problem
- Randomized Quicksort
- Moments and Deviations
- Chernoff Bounds and Hoeffding Bounds
- Balls and Bins
- Continuous Distributions and the Poisson Process
- Markov Chains and Random Walks
- Monte Chains Monte Carlo (MCMC)

Grading Policy

- Attendance (10%)
- Assignments (x 6, 30%)
- Midterm Presentation (30%)
- Final Presentation (30%)

Traditional deterministic algorithms

Randomized algorithms

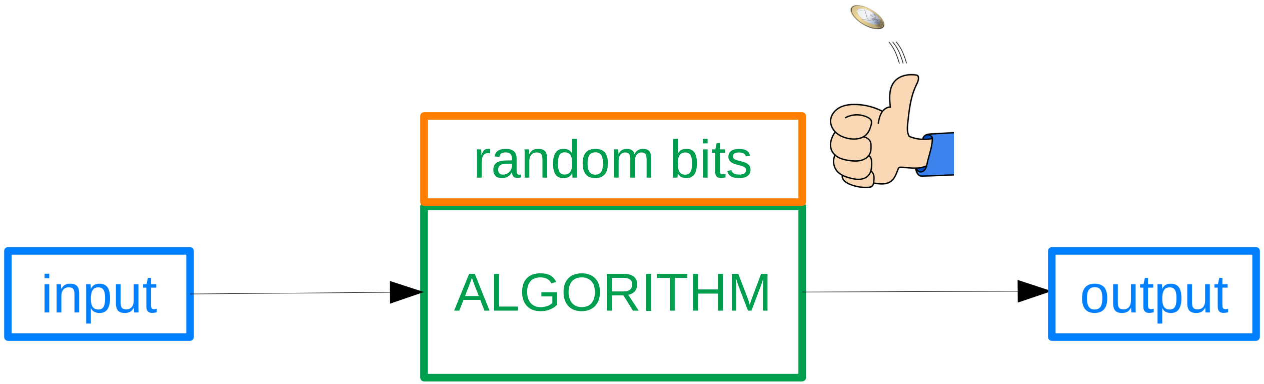


figure freely downloaded from https://pixabay.com/illustrations/coin-flipping-coin-hand-flip-flick-5822271/

Why?

- Randomized algorithms are
  - often much ***simpler*** than the best known deterministic ones.
  - often much ***more efficient*** (faster or using less space) than the best known deterministic ones.

- Sometimes ideas from the randomized algorithms lead to good deterministic algorithms.

Comparisons

- It's different from the *average-case* analysis of deterministic algorithms.
  - e.g., expected running time of a deterministic algorithm on input sampled from a distribution.

- In most scenarios, it's NOT a heuristic algorithm.
  - The accuracy is guaranteed, or
  - The running time is guaranteed.

An illustrating example:

- **Problem:** find a grade-‘*A*’ student in a class of *n* students where half of them get ‘A’.

- What is the time complexity for the best deterministic algorithm?
  - I mean, in the “worst case”.

A randomized algorithm (from Wikipedia)

```
findingA_LV(array L, n)
begin
    repeat
        Randomly select one element out of n elements.
    until 'A' is found
end
```

Assignment: Prove that the expected number of iterations is $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{i}{2^i}\le 2.$

A randomized algorithm (from Wikipedia)

```
findingA_MC(array L, n, k)
begin
    i ← 0
    repeat
        Randomly select one element out of n elements.
        i ← i + 1
    until i = k or 'A' is found
end
```

After k iterations, $\Pr[\text{find } A] = 1 - (1/2)^k$.

Birthday problem (paradox)

- There are n randomly chosen people in a room.
- How ***large*** should n be such that there is at least one pair of them having the same birthday (mm/dd)?
- By the pigeonhole principle, $n = 367$? or 366?

- Let us consider this problem in the other way around.

  How ***large*** should n be such that there is at least one pair of them having the same birthday (mm/dd) with probability ≥ 0.5?

Birthday problem (paradox)

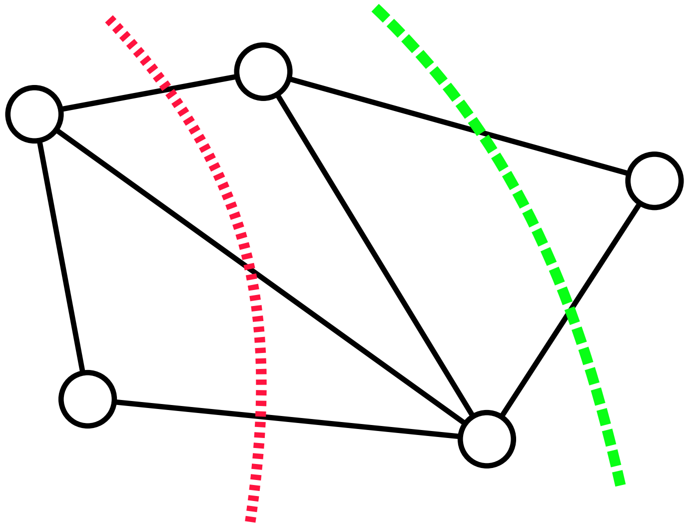
- n people: $x_1, x_2, \ldots, x_n$
- Event i: some pair of $x_1, x_2, \ldots, x_i$ have the same birthday.
- $\Pr[\text{Event2}] = 1 - \frac{364}{365}$
- $\Pr[\text{Event3}] = 1 - \frac{364}{365} \cdot \frac{363}{365}$
- ...
- $\Pr[\text{Event23}] = 1 - \frac{364}{365} \cdot \frac{363}{365} \cdots \frac{343}{365} \approx 0.507297.$

- 23 is much less than 366 or 367.

Birthday problem (paradox)

- **Assignment:** Compute n such that there is at least one pair of them having the same birthday with probability ≥ 0.9.

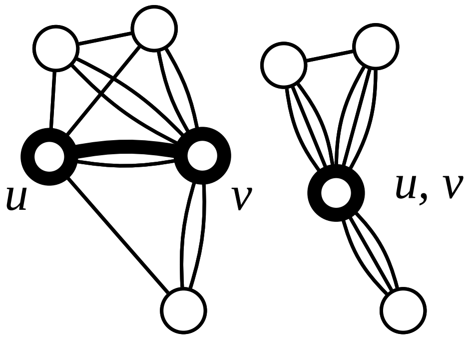
Min-Cut



$|V| = n, |E| = m$

- A graph G = (V, E) and its two "cuts".
  - **Cut**: a partition of the vertices in V into two non-empty, disjoint sets S and T such that
  - $S \cup T = V$
- The **cutset** of a cut:
  - $\{uv \in E \mid u \in S, v \in T\}$.
- The size of the cut:
  - the cardinality of its cutset.

Edge contraction



$e = (u, v)$

$G \rightarrow G/e$

Karger's edge-contraction algorithm (1993)

Procedure contract ($G = (V, E)$):

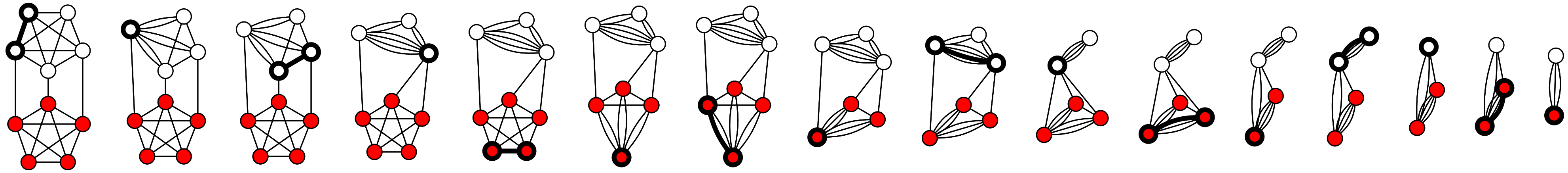
while $|V| > 2$:

  choose $e \in E$ uniformly at random

  $G \leftarrow G/e$

return the only cut in G

Time complexity: $O(m)$ or $O(n^2)$.



Analysis

- C: a specific cut of G.
- k: the size of the cut C.
- The minimum degree of G must be $\geq k$. (WHY?)
  - So, $|E| \geq nk/2$.
- The probability that the algorithm picks an edge from C to contract is

$$\frac{k}{|E|} \leq \frac{k}{nk/2} = \frac{2}{n}.$$

Analysis (contd.)

- Let p_n be the probability that the algorithm on an n-vertex graph avoids C.
- Then,

$$p_n \geq \left(1 - \frac{2}{n}\right) \cdot p_{n-1}$$

- The recurrence can be expanded as

$$p_n \geq \prod_{i=0}^{n-3} \left(1 - \frac{2}{n-i}\right) = \frac{n-2}{n} \cdot \frac{n-3}{n-1} \cdots \frac{2}{4} \cdot \frac{1}{3} = \frac{1}{\binom{n}{2}}.$$

Analysis (contd.)

- The probability of "success" is $\binom{n}{2}^{-1}$
  - A bit too low, isn't it?

- How about repeating it for $T = \binom{n}{2} \ln n$ times, and then choose the minimum of them?

- The probability of NOT finding a min-cut is $\left[1 - \binom{n}{2}^{-1}\right]^T \le \frac{1}{e^{\ln n}} = \frac{1}{n}.$

- Total running time: $O(Tm)$ or $O(Tn^2)$.

A bonus project (5%)

- Implement Karger's edge-contraction algorithm.

Assignment 1 (5%)

A randomized algorithm (from Wikipedia) (3%)

```
findingA_LV(array L, n)
begin
   repeat
      Randomly select one element out of n elements.
   until 'A' is found
end
```

Assignment: Prove that the expected number of iterations is $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{i}{2^i} \leq 2.$

Birthday problem (paradox) (2%)

- **Assignment:** Compute *n* such that there is at least one pair of them having the same birthday with probability ≥ 0.9.