Randomized Algorithms

Chernoff and Hoeffding Bounds

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Moment Generating Functions

- Definition. The moment generating function of a random variable X is

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$\mathbf{E}[X^n] = M_X^{(n)}(0)$ The nth derivative of $M_X(t)$ at $t = 0$.

Example

- Consider a geometric random variable X with parameter p.
- For $t < -\ln(1-p)$,

$$\begin{aligned} M_X(t) &= \mathbf{E}[e^{tX}] \\ &= \sum_{k=1}^{\infty}(1-p)^{k-1}pe^{tk} \\ &= \frac{p}{1-p}\sum_{k=1}^{\infty}(1-p)^k e^{tk} \\ &= \frac{p}{1-p}((1-(1-p)e^t)^{-1}-1). \end{aligned}$$

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\end{aligned}
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$$\therefore M_X^{(1)}(t) = p(1-(1-p)e^t)^{-2} e^t,$$

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M_X^{(2)}(t) &= 2p(1-p)(1-(1-p)e^t)^{-3} e^{2t} \\
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$$\mathbf{E}[X] = M_X^{(1)}(0) = \frac{1}{p},\quad \mathbf{E}[X^2] = M_X^{(2)}(0) = \frac{(2-p)}{p^2}.$$

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- *Proof.*

$$M_{X+Y}(t) = \mathbf{E}[e^{t(X+Y)}] = \mathbf{E}[e^{tX}e^{tY}] = \mathbf{E}[e^{tX}] \cdot \mathbf{E}[e^{tY}] = M_X(t) \cdot M_Y(t).$$

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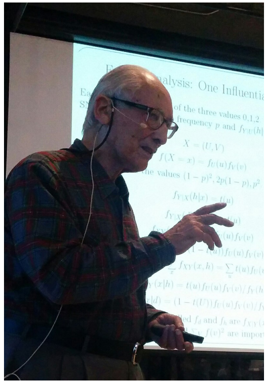
- *Generalization:*

$$M_{X_1+X_2+\cdots+X_k}(t) = M_{X_1}(t) \cdot M_{X_2}(t) \cdots M_{X_k}(t).$$

Chernoff bounds: Applying Markov's inequality to e^{tX}



Herman Chernoff
https://en.wikipedia.org/wiki/Herman_Chernoff

- From Markov's inequality,

For any $t > 0$,

$$\Pr[X \geq a] = \Pr[e^{tX} \geq e^{ta}] \leq \frac{\mathbf{E}[e^{tX}]}{e^{ta}}.$$

$$\Pr[X \geq a] \leq \min_{t>0} \frac{\mathbf{E}[e^{tX}]}{e^{ta}}.$$

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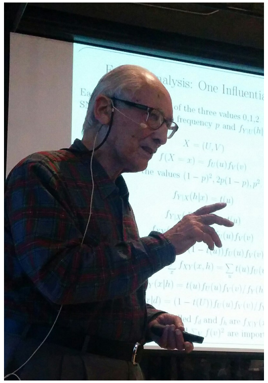

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- Choose appropriate values for t for specific distributions.

Chernoff bounds for sum of Poisson trials

- Poisson trials:

  ≈ Bernoulli trials

  - while the trials are **not necessarily identical**.
- $X_1, \ldots, X_n$: independent Poisson trials with $\Pr[X_i = 1] = p_i$.
- Let $X = \sum_{i=1}^{n} X_i$

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$$\mu = \mathbf{E}[X] = \mathbf{E}\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \mathbf{E}[X_i] = \sum_{i=1}^{n} p_i .$$

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$$\begin{aligned} M_{X_i}(t) &= \mathbf{E}[e^{tX_i}] \\ &= p_i e^t + (1-p_i)e^0 \\ &= 1 + p_i(e^t - 1) \\ &\leq e^{p_i(e^t-1)}. \end{aligned}$$

$$\begin{aligned} \therefore M_X(t) &= \prod_{i=1}^{n} M_{X_i}(t) \\ &\leq \prod_{i=1}^{n} e^{p_i(e^t-1)} \\ &= \exp\left\{\sum_{i=1}^{n} p_i(e^t-1)\right\} \\ &= e^{(e^t-1)\mu}. \end{aligned}$$

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- Theorem. Let $X_1, \ldots, X_n$ be independent Poisson trials with $\Pr[X_i = 1] = p_i$. Let $X = \sum_{i=1}^{n} X_i$ and $\mu = \mathbf{E}[X]$. Then the following Chernoff bounds hold:

1. For $\delta > 0$,
$$\Pr[X \geq (1+\delta)\mu] \leq \left(\frac{e^{\delta}}{(1+\delta)^{(1+\delta)}}\right)^{\mu};$$
2. For $0 < \delta \leq 1$,
$$\Pr[X \geq (1+\delta)\mu] \leq e^{-\mu\delta^2/3};$$
3. For $R \geq 6\mu$,
$$\Pr[X \geq R] \leq 2^{-R}.$$

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- Applying Markov's inequality for $t > 0$,

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$$f(\delta) := \delta - (1+\delta)\ln(1+\delta) + \frac{\delta^2}{3} \leq 0$$

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- Taking logarithm of both sides:

$$\begin{aligned}f(\delta) &:= \delta - (1+\delta)\ln(1+\delta) + \frac{\delta^2}{3} \leq 0 \\ f'(\delta) &= -\ln(1+\delta) + \frac{2}{3}\delta, \\ f''(\delta) &= -\frac{1}{1+\delta} + \frac{2}{3}.\end{aligned}$$

Proof sketch

- Applying Markov's inequality for $t > 0$,

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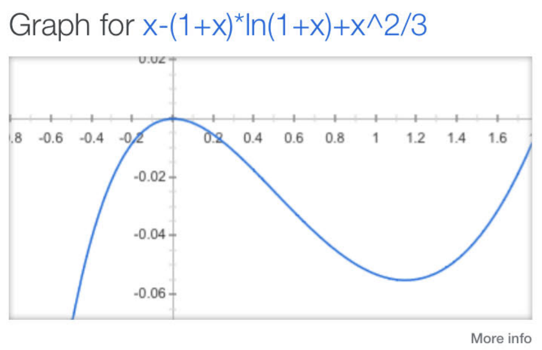

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- <u>Theorem</u>. Let $X_1, \ldots, X_n$ be independent Poisson trials with $\Pr[X_i = 1] = p_i$.
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$$\Pr[X \le (1-\delta)\mu] \le \left(\frac{e^{\delta}}{(1-\delta)^{(1-\delta)}}\right)^{\mu}.$$

$$\Pr[X \le (1-\delta)\mu] \le e^{-\mu\delta^2/2}.$$

  ✔ Therefore we have:

$$\Pr[|X-\mu| \ge \delta\mu] \le 2e^{-\mu\delta^2/3}.$$

Example: 75% heads in fair coin flips

- $$X_i = \begin{cases} 1 & \text{if the } i\text{th coin flip is head} \\ 0 & \text{otherwise} \end{cases}$$

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- Try to use Chernoff bound!

$$\begin{aligned} \Pr\left[\left|X - \frac{n}{2}\right| \geq \frac{n}{4}\right] &\leq 2\exp\left\{-\frac{1}{3}\frac{n}{2}\left(\frac{1}{2}\right)^2\right\} \\ &\leq 2e^{-n/24}. \end{aligned}$$

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$$\mathbf{Var}[X_i] = \mathbf{E}[X_i^2] - (\mathbf{E}[X_i])^2 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}.$$

$$\mathbf{Var}[X] = \mathbf{Var}\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \mathbf{Var}[X_i] = \frac{n}{4}.$$

$$\Pr[X \geq 3n/4] \leq \Pr[|X - \mathbf{E}[X]| \geq n/4] \leq \frac{\mathbf{Var}[X]}{(n/4)^2} = \frac{n/4}{(n/4)^2} = \frac{4}{n}.$$

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	n=50	n=100	n=200
$2/n$	0.2	0.02	0.01
$e^{-n/24}$	0.125	0.016	0.00025

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- Such a device is somehow too weak to be used in practical.
- Let say we run the device for n = 201 times for each examination and output "True" if more than 101 of the results reveal that the diamond is real and output "False" otherwise. **(majority vote)**

$$\Pr[X \leq n/2] = \Pr\left[X - \frac{2n}{3} \leq -\frac{n}{6}\right] \leq e^{-(2n/3)\cdot(1/4)^2\cdot(1/2)} = e^{-n/48} < 0.016.$$

X_i : 1 if ith test is correct and 0 otherwise

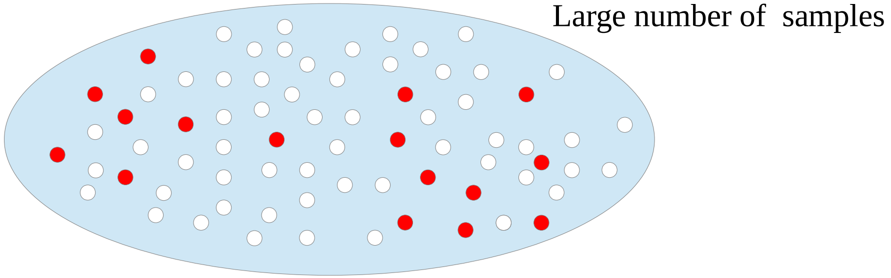
An application: Parameter Estimation

- **Goal:** evaluating the probability that a particular gene mutation occurs in the population.
- A lab test can determine if a DNA sample carries the mutation.
- However, the test is very **expensive**, so we want to obtain a relatively reliable estimate from a **small** number of samples.

https://tinyurl.com/8fwxsnab

An application: Parameter Estimation

- *p*: be the unknown value we try to estimate.
- *n*: the number of samples we have
- $X = \tilde{p}n$: number of samples having the mutation



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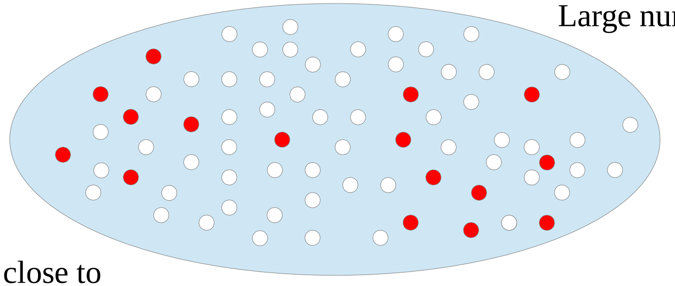


We expect the value p to be close to the sampled value $\tilde{p}$

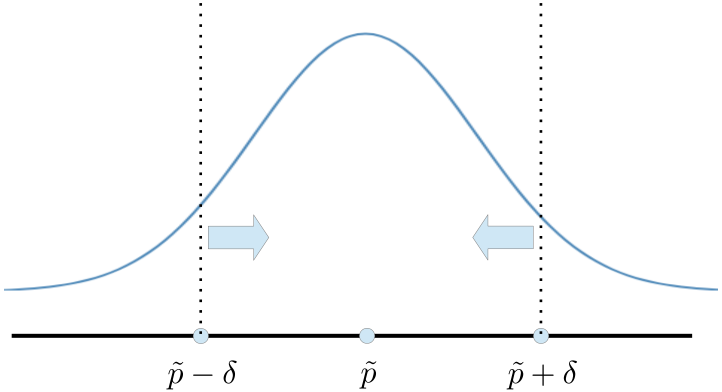
An application: Parameter Estimation

- Definition. A $1-\gamma$ **confidence interval** for a parameter p is an interval

$$[\tilde{p}-\delta,\ \tilde{p}+\delta]$$

such that

$$\Pr[p \in [\tilde{p}-\delta,\ \tilde{p}+\delta]] \geq 1-\gamma.$$



We need to find values of δ and γ such that

$$\Pr[p \in [\tilde{p}-\delta,\ \tilde{p}+\delta]] = \Pr[np \in [n(\tilde{p}-\delta),\ n(\tilde{p}+\delta)]] \geq 1-\gamma.$$

An application: Parameter Estimation

- Apply the Chernoff bound:

$$\begin{aligned}\Pr[p \notin [\tilde{p} - \delta,\ \tilde{p} + \delta]] &= \Pr\left[X < np\left(1 - \frac{\delta}{p}\right)\right] + \Pr\left[X > np\left(1 + \frac{\delta}{p}\right)\right] \\ &< e^{-np(\delta/p)^2/2} + e^{-np(\delta/p)^2/3} \\ &= e^{-n\delta^2/2p} + e^{-n\delta^2/3p}.\end{aligned}$$

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Setting $\gamma = e^{-n\delta^2/2} + e^{-n\delta^2/3}$, we obtain a trade-off between δ and n.

Example

- Set $\gamma = 0.05$, $\delta = 0.03$.

$$
\begin{aligned}
& e^{-n(0.03)^2/2} + e^{-n(0.03)^2/3} < 2e^{-n(0.03)^2/3} < 0.05 \\
\Rightarrow \quad & e^{-n(0.03)^2/3} < 0.025 \\
\Rightarrow \quad & -n(0.03)^2/3 < \ln(0.025) \approx -3.6889 \\
\Rightarrow \quad & n > 3.6889 \cdot 3/(0.03)^2 \approx 12296.33.
\end{aligned}
$$

The Hoeffding Bound

Wassily Hoeffding (1914–1991)

refer to https://tinyurl.com/mzz7x8pb

- Extending the Chernoff bound to general random variables with a **bounded range**.

Hoeffding's Lemma: Let X be a random such that $\Pr[X \in [a, b]] = 1$ and $\mathbf{E}[X] = 0$. Then for every $\lambda > 0$,

$$\mathbf{E}[e^{\lambda X}] \le e^{\lambda^2 (b-a)^2/8}.$$

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Hoeffding's Bound: Let $X_1, \ldots, X_n$ be independent random variables such that for all $1 \le i \le n$, Then for every $\lambda > 0$, $\mathbf{E}[X_i] = \mu$ and $\Pr[a \le X_i \le b] = 1$. Then

$$\Pr\left[\left|\frac{1}{n}\sum_{i=1}^{n} X_i - \mu\right| \ge \epsilon\right] \le 2e^{-2n\epsilon^2/(b-a)^2}.$$

The Hoeffding Bound

Wassily Hoeffding (1914−1991)

refer to https://tinyurl.com/mzz7x8pb

- Extending the Chernoff bound to general random variables with a **bounded range**.

Theorem: Let $X_1, \ldots, X_n$ be independent random variables such that $\mathbf{E}[X_i] = \mu_i$ and $\Pr[a_i \le X_i \le b_i] = 1$ for constant a_i and b_i. Then,

$$\Pr\left[\left|\sum_{i=1}^{n} X_i - \sum_{i=1}^{n} \mu_i\right| \ge \epsilon\right] \le 2e^{-2\epsilon^2/\sum_{i=1}^{n}(b_i - a_i)^2}.$$

Proofs

Proof of Hoeffding's Lemma

Hoeffding's Lemma: Let X be a random such that $\Pr[X \in [a, b]] = 1$ and $\mathbf{E}[X] = 0$. Then for every $\lambda > 0$,

$$\mathbf{E}[e^{\lambda X}] \leq e^{\lambda^2 (b-a)^2/8}.$$

- We assume $a < 0$ and $b > 0$. (Why?)
- $f(x) = e^{\lambda x}$ is a convex function.

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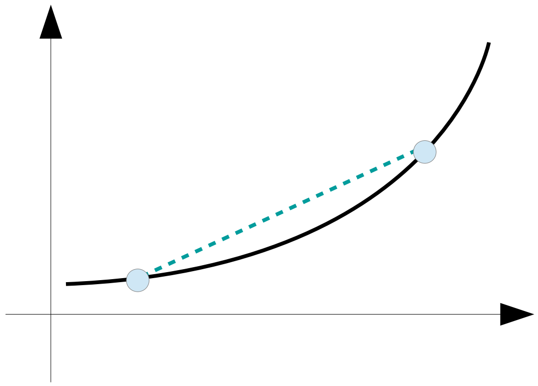
$$f(\alpha a + (1-\alpha) b) \le \alpha e^{\lambda a} + (1-\alpha) e^{\lambda b}.$$



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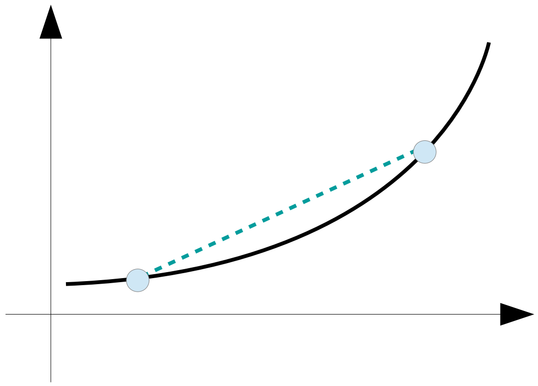
$$\text{For } x \in [a,b], \text{ let } \alpha = \frac{b-x}{b-a}, \text{ then } x = \alpha a + (1-\alpha) b$$



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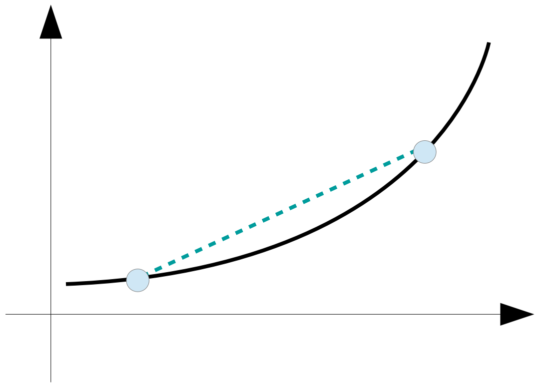
For $x \in [a,b]$, let $\alpha = \dfrac{b-x}{b-a}$, then $x = \alpha a + (1-\alpha) b$

$$\therefore e^{\lambda x} \le \frac{b-x}{b-a} e^{\lambda a} + \frac{x-a}{b-a} e^{\lambda b}.$$



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$$\begin{aligned}
\mathbf{E}[e^{\lambda X}] &\le \mathbf{E}\left[\frac{b-X}{b-a}e^{\lambda a}\right] + \mathbf{E}\left[\frac{X-a}{b-a}e^{\lambda b}\right] \\
&= \frac{b}{b-a}e^{\lambda a} - \frac{\mathbf{E}[X]}{b-a}e^{\lambda a} - \frac{a}{b-a}e^{\lambda b} + \frac{\mathbf{E}[X]}{b-a}e^{\lambda b} \\
&= \frac{b}{b-a}e^{\lambda a} + \frac{a}{b-a}e^{\lambda b}.
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&= \frac{b}{b-a}e^{\lambda a} + \frac{a}{b-a}e^{\lambda b} \\
&= e^{\lambda a}\left(\frac{b}{b-a} - \frac{a}{b-a}e^{\lambda(b-a)}\right).
\end{aligned}$$

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&= e^{\lambda a}(1 - \theta + \theta e^{\lambda(b-a)}) \\
&= e^{-\theta\lambda(b-a)}(1 - \theta + \theta e^{\lambda(b-a)}).
\end{aligned}$$

let $\theta = \dfrac{-a}{b-a}$

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&= e^{-\theta\lambda(b-a)}(1 - \theta + \theta e^{\lambda(b-a)}) \\
&= e^{\phi(\lambda(b-a))}.
\end{aligned}$$

Let $\phi(t) = -\theta t + \ln(1 - \theta + \theta e^t)$.

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$$\phi(t) = \phi(0) + t\phi'(0) + \frac{1}{2}t^2\phi''(t') \leq \frac{1}{8}t^2.$$

← Taylor's theorem, $\forall t > 0, \exists t' \in [0, t]$

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For $t = \lambda(b-a)$,

$$\phi(\lambda(b-a)) \le \frac{\lambda^2(b-a)^2}{8}.$$

Thus, $\mathbf{E}[e^{\lambda X}] \le e^{\phi(\lambda(b-a))} \le e^{\lambda^2(b-a)^2/8}$.

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Hoeffding's Bound: Let $X_1, \ldots, X_n$ be independent random variables such that for all $1 \le i \le n$, Then for every $\lambda > 0$, $\mathbf{E}[X_i] = \mu$ and $\Pr[a \le X_i \le b] = 1$. Then

$$\Pr\left[\left|\frac{1}{n}\sum_{i=1}^{n} X_i - \mu\right| \ge \epsilon\right] \le 2e^{-2n\epsilon^2/(b-a)^2}.$$

Let $Z_i = X_i - \mathbf{E}[X_i]$ and $Z = \frac{1}{n}\sum_{i=1}^{n} Z_i$.

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$$\begin{aligned}\Pr[Z \ge \epsilon] = \Pr[e^{\lambda Z} \ge e^{\lambda\epsilon}] \le \frac{\mathbf{E}[e^{\lambda Z}]}{e^{\lambda\epsilon}} &= \frac{\prod_{i=1}^{n} \mathbf{E}[e^{\lambda Z_i/n}]}{e^{\lambda\epsilon}} \\ &\le \frac{\prod_{i=1}^{n} e^{\lambda^2(b-a)^2/(8n^2)}}{e^{\lambda\epsilon}} \\ &\le \frac{e^{\lambda^2(b-a)^2/8n}}{e^{\lambda\epsilon}}.\end{aligned}$$

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Note: $Z_i/n \in [(a-\mu)/n, (b-\mu)/n]$.

Setting $\lambda = \frac{4n\epsilon}{(b-a)^2}$,

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Let $Z_i = X_i - \mathbf{E}[X_i]$ and $Z = \frac{1}{n}\sum_{i=1}^{n} Z_i$.

$$\Pr\left[\frac{1}{n}\sum_{i=1}^{n} X_i - \mu \ge \epsilon\right] = \Pr[Z \ge \epsilon] \le e^{-2n\epsilon^2/(b-a)^2}.$$

For $\Pr[Z \le -\epsilon]$ with $\lambda = -\frac{4n\epsilon}{(b-a)^2}$

$$\Pr\left[\frac{1}{n}\sum_{i=1}^{n} X_i - \mu \le -\epsilon\right] = \Pr[Z \le -\epsilon] \le e^{-2n\epsilon^2/(b-a)^2}.$$

Exercise

Consider a collection $X_1, \ldots, X_n$ of n independent integers chosen uniformly from the set $\{0, 1, 2\}$.

Let $X = \sum_{i=1}^{n} X_i$ and $0 < \delta < 1$.

Derive a Chernoff bound for $\Pr[X \geq (1+\delta)n]$ and $\Pr[X \geq (1-\delta)n]$.

Exercise

In an election with two candidates using paper ballots, each vote is independently misreported with probability $p = 0.02$.

Use a Chernoff bound to give an upper bound on the probability that more than 4% of the votes are misreported in an election of 1,000,000 ballots.