# Randomized Algorithms

## Balls and Bins: Some basics

Joseph Chuang-Chieh Lin
Dept. CSIE, Tamkang University

# Outline

- The Birthday Paradox – Revisited
- Balls into Bins
- Poisson Distribution

# Exponential function - revisited

- By definition of $e^x$:

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots$$

$$\Rightarrow \quad e = \sum_{k=0}^{\infty} \frac{1}{k!} = 1 + x + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} \cdots$$

# Exponential function - revisited

- By the Binomial Theorem:

$$\begin{aligned}\left(1+\frac{1}{n}\right)^n &= \sum_{k=0}^{n}\binom{n}{k}\left(\frac{1}{n}\right)^k \\ &= \sum_{k=0}^{n}\frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}\cdot\frac{1}{n^k} \\ &= \sum_{k=0}^{n}\left(\frac{n}{n}\right)\cdot\left(\frac{n-1}{n}\right)\cdots\left(\frac{n-k+1}{n}\right)\cdot\frac{1}{k!}.\end{aligned}$$

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&= \sum_{k=0}^{n} \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!} \cdot \frac{1}{n^k} \\
&= \sum_{k=0}^{n} \left(\frac{n}{n}\right) \cdot \left(\frac{n-1}{n}\right) \cdots \left(\frac{n-k+1}{n}\right) \cdot \frac{1}{k!}.
\end{aligned}
$$

$$
\therefore \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = \sum_{k=0}^{\infty} \frac{1}{k!}.
$$

# Exponential function - revisited

- Useful approaches:

$$\left(1+\frac{1}{n}\right)^{-m} = \left(\left(1+\frac{1}{n}\right)^{n}\right)^{-m/n} \approx e^{-m/n}.$$

$$\frac{k^k}{k!} < \sum_{i=0}^{\infty} \frac{k^i}{i!} = e^k.$$

$$\therefore k! > \left(\frac{k}{e}\right)^k.$$

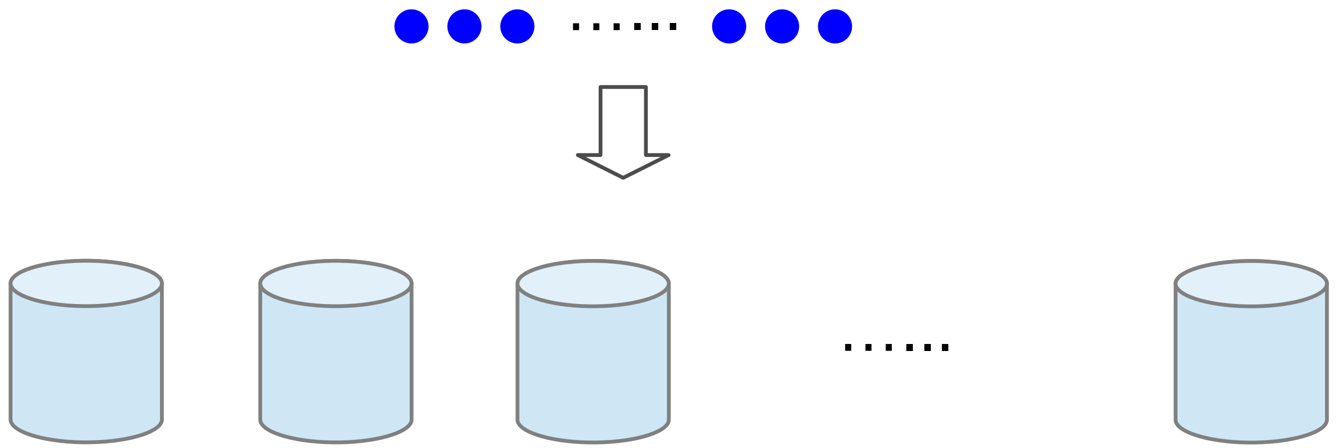
# Exercise

- Show that for $|x| \leq 1$, $e^x(1-x^2) \leq 1+x \leq e^x$.

- Let $\left(1-\left(1-\frac{1}{n}\right)^{km}\right)^k \approx (1-e^{\square})^k$.

  Find $\square$ = ?

# *m* balls into *n* bins



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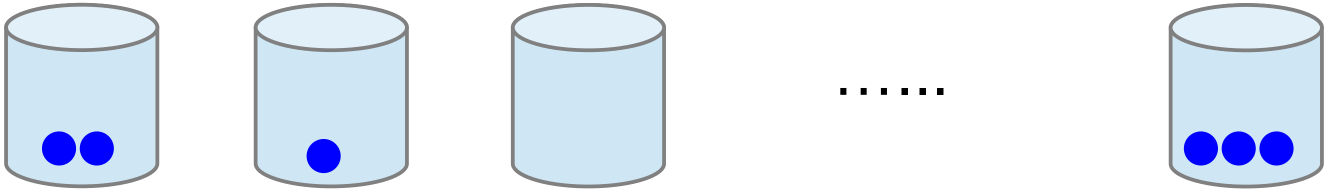

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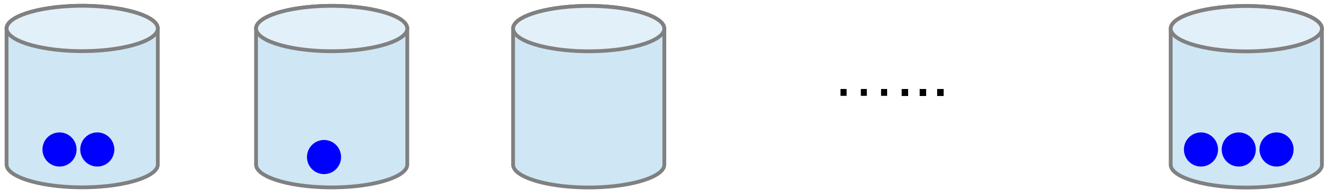

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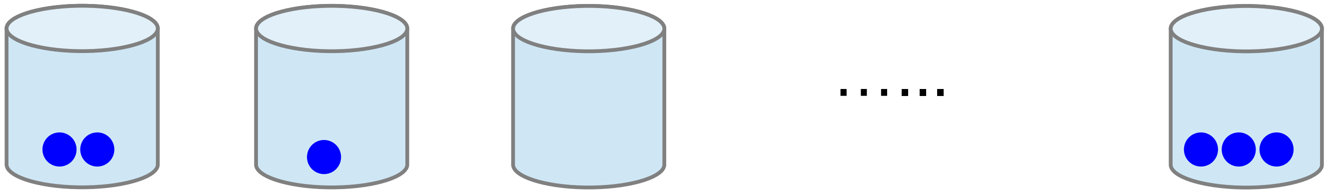
- Let's say there are 30 people in a room. What's the probability that no two people in the room share the same birthday?



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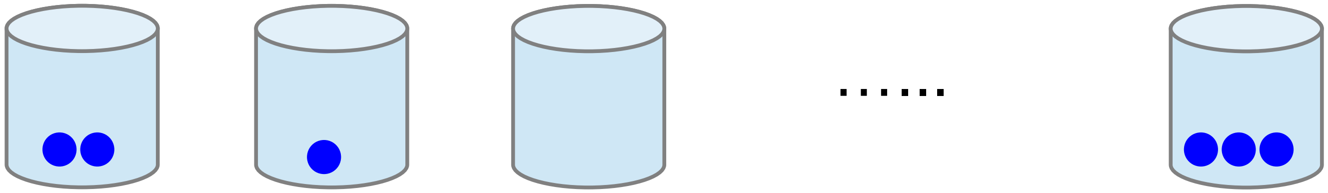


$$\frac{\binom{365}{30}30!}{365^{30}}.$$

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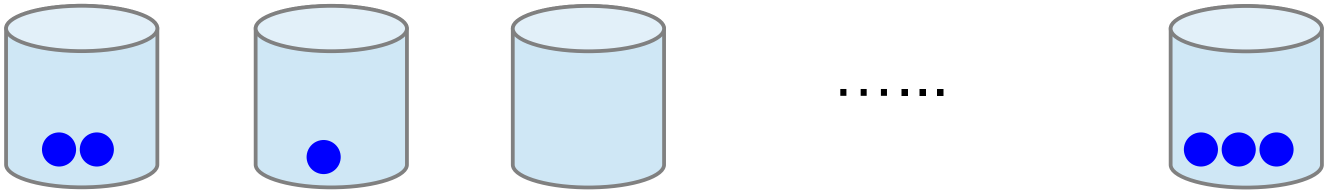


$$\left(1-\frac{1}{365}\right)\cdot\left(1-\frac{2}{365}\right)\cdots\left(1-\frac{29}{365}\right).$$

# *m* balls into *n* bins

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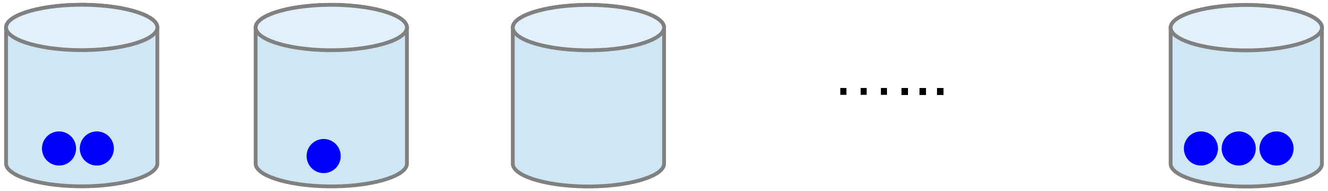


$$\left(1-\frac{1}{n}\right)\cdot\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{m-1}{n}\right)=\prod_{j=1}^{m-1}\left(1-\frac{j}{n}\right).$$

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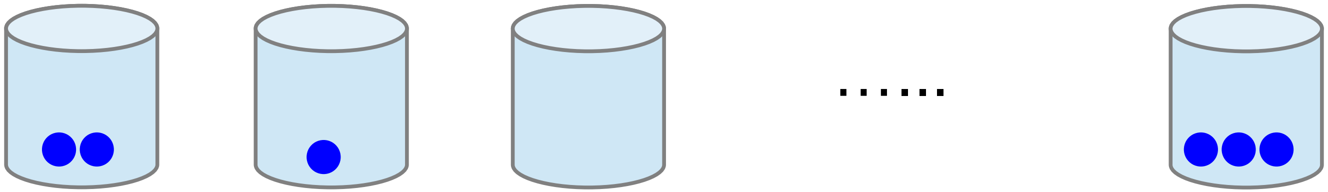


$$\left(1-\frac{1}{n}\right)\cdot\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{m-1}{n}\right)=\prod_{j=1}^{m-1}\left(1-\frac{j}{n}\right)\approx\prod_{j=1}^{m-1}e^{-j/n}=\exp\left\{-\sum_{j=1}^{m-1}\frac{j}{n}\right\}=e^{-m(m-1)/2n}\approx e^{-m^2/2n}.$$

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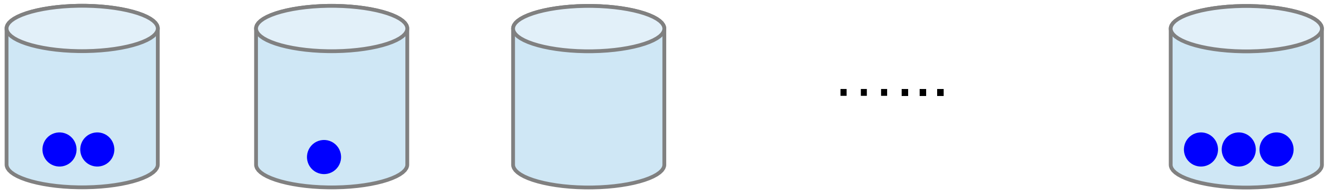

- Set the probability threshold to be ½:

$$\frac{m^2}{2n} = \ln 2 \Rightarrow m = \sqrt{2n \ln 2} \approx 22.49.$$

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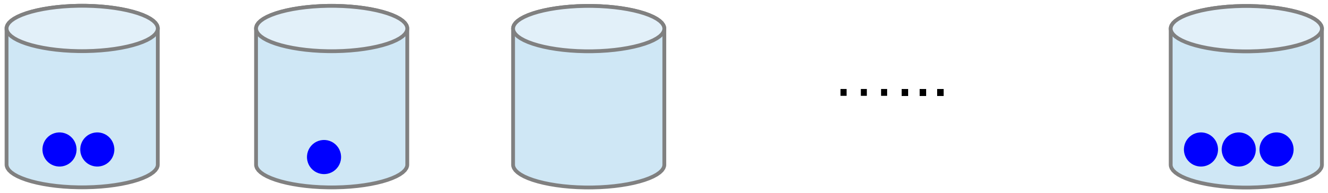

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Matching your observation?

# ***n*** balls into *n* bins: maximum load

- ***n*** balls are thrown independently and uniformly at random into *n* bins.
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    - The average load is $n/n = 1$?!

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**union bound**

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n\left(\frac{e}{M}\right)^M &\le n\left(\frac{e\ln\ln n}{3\ln n}\right)^{3\ln n/\ln\ln n} \\
&\le n\left(\frac{\ln\ln n}{\ln n}\right)^{3\ln n/\ln\ln n} = e^{\ln n}\left(\frac{e^{\ln\ln\ln n}}{e^{\ln\ln n}}\right)^{3\ln n/\ln\ln n} \\
&= e^{\ln n}(e^{\ln\ln\ln n-\ln\ln n})^{3\ln n/\ln\ln n} \\
&= e^{-2\ln n+3(\ln n)(\ln\ln n)/\ln\ln n} \\
&\le \frac{1}{n}.
\end{aligned}$$

# Application: Bucket Sort

- A set of $n = 2^m$ numbers chosen uniformly at random in $[0, 2^k)$, $k \geq m$.

**Bucket Sort:**

– Stage 1: place the elements into $n$ buckets.
  - $j^{th}$ bucket: holds all elements whose first $m$ binary digits corresponds to $j$.
    – e.g., $n = 2^{10}$, bucket 3 contains all elements whose first 10 digits are 0000000011.

– Stage 2: sort each bucket using any standard sorting algorithm.

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**Bucket Sort:** Expected $O(n)$ time?!

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  - The number of elements landing in a specific bucket: Binomial($n$, $1/n$).
- Stage 2: sort each bucket using any standard sorting algorithm.

# Application: Bucket Sort

- $X_j$: the number of elements landing in bucket $j$.
- The time to sort bucket $j$: $c(X_j)^2$, for some constant $c$.
- The expected time for sorting in Stage 2:

$$\mathbf{E}\left[\sum_{j=1}^{n} c(X_j)^2\right] = c\sum_{j=1}^{n}\mathbf{E}[X_j^2] = cn\mathbf{E}[X_1^2].$$

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The same for all buckets.

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$$\begin{aligned}
\mathbf{E}[X_1^2] &= \mathbf{Var}[X_1] + (\mathbf{E}[X_1])^2 \\
&= n \cdot \frac{1}{n}\left(1 - \frac{1}{n}\right) + \left(n \cdot \frac{1}{n}\right)^2 \\
&= 2 - \frac{1}{n} \\
&< 2.
\end{aligned}$$

# Assignment 05

- Suppose that $n$ balls are thrown independently and uniformly at random into $n$ bins.
  - Find the conditional *probability* that bin 1 has one ball given that exactly one ball fell into the first three bins.
  - Find the conditional expectation of the number of balls in bin 1 under the condition that bin 2 received no balls.

# Questions

- What is the probability that **a given bin is empty**?
- What is the expected number of **empty bins**?

- The probability that the $i$th bin remains empty is

$$\left(1 - \frac{1}{n}\right)^m \approx e^{-m/n}.$$

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The expected "fraction"

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$$
\begin{aligned}
p_r &= \binom{m}{r}\left(\frac{1}{n}\right)^r\left(1-\frac{1}{n}\right)^{m-r} \\
&= \frac{1}{r!}\frac{m(m-1)\cdots(m-r+1)}{n^r}\left(1-\frac{1}{n}\right)^{m-r}. \\
&\approx \frac{e^{-m/n}(m/n)^r}{r!}.
\end{aligned}
$$

Expected number of bins with exactly *r* balls $\approx np_r$.

# The Poisson Distribution

Siméon Poisson
(1781−1840)
Wikipedia

- A discrete Poisson random variable $X$ with parameter $\mu$ is given by the following probability distribution on $j = 0, 1, 2, \ldots$:

$$\Pr[X = j] = \frac{e^{-\mu}\mu^j}{j!}.$$

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$$\updownarrow$$

$$\frac{e^{-m/n}(m/n)^r}{r!}.$$

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- Try to verify if it's proper:

$$\sum_{j=0}^{\infty} \Pr[X = j] = \sum_{j=0}^{\infty} \frac{e^{-\mu}\mu^j}{j!}$$
$$= e^{-\mu} \sum_{j=0}^{\infty} \frac{\mu^j}{j!}$$
$$= 1.$$

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- The expectation:

$$
\begin{aligned}
\mathbf{E}[X] &= \sum_{j=0}^{\infty} j \Pr[X = j] = \sum_{j=1}^{\infty} j \frac{e^{-\mu}\mu^j}{j!} \\
&= \mu \cdot \sum_{j=1}^{\infty} \frac{e^{-\mu}\mu^{j-1}}{(j-1)!} \\
&= \mu \cdot \sum_{j=0}^{\infty} \frac{e^{-\mu}\mu^{j}}{j!} \\
&= \mu.
\end{aligned}
$$

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$$
\begin{aligned}
\Pr[X+Y=j] &= \sum_{k=0}^{j} \Pr[(X=k)\cap(Y=j-k)] \\
&= \sum_{k=0}^{j} \frac{e^{-\mu_1}\mu_1^k}{k!}\frac{e^{-\mu_2}\mu_2^{(j-k)}}{(j-k)!} \\
&= \frac{e^{-(\mu_1+\mu_2)}}{j!}\sum_{k=0}^{j}\frac{j!}{k!(j-k)!}\mu_1^k\mu_2^{(j-k)} \\
&= \frac{e^{-(\mu_1+\mu_2)}}{j!}\sum_{k=0}^{j}\binom{j}{k}\mu_1^k\mu_2^{(j-k)} \\
&= \frac{e^{-(\mu_1+\mu_2)}(\mu_1+\mu_2)^j)}{j!}.
\end{aligned}
$$

- The sum of finite number of independent Poisson random variables is a Poisson random variable.

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For any $t$,

$$\mathbf{E}[e^{tX}] = \sum_{k=0}^{\infty} \frac{e^{-\mu}\mu^k}{k!} e^{tk} = e^{\mu(e^t-1)} \sum_{k=0}^{\infty} \frac{e^{-\mu e^t}(\mu e^t)^k}{k!}$$

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$$= \quad e^{\mu(e^t-1)}.$$

$$M_{X+Y}(t) = M_X(t) \cdot M_Y(t) = e^{(\mu_1+\mu_2)(e^t-1)}.$$

# Limit of the Binomial Distribution

- <u>Theorem.</u> Let $X_n \sim$ Binomial($n$, $p$) be a binomial random variable
  - $p$: a function of $n$
  - $\lim_{n\to\infty} np = \lambda$ is a constant, independent of $n$.

  Then for any fixed $k$,

$$\lim_{n\to\infty} \Pr[X_n = k] = \frac{e^{-\lambda}\lambda^k}{k!}.$$

# Scenario

- *m* balls into *n* bins.
  - $m = f(n)$;
  - $\lim_{m\to\infty} m/n = \lambda$;
  - $X_m$: the number of balls in a specific bin.
    - Binomial($m$, $1/n$).

- From the theorem:

$$\lim_{m\to\infty} \Pr[X_m = r] = \frac{e^{-m/n}(m/n)^r}{r!}.$$