# Celebrity Games

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# Outline

# Celebrity games

- A new model of network creation games.
  - Players have weights.
  - There is a critical distance $\beta$.
  - The cost incurred by a player:
    - ⋆ Establishing links;
    - ⋆ Sum of weights of players farther than the critical distance.
- The celebrity game vs. the network creation game.
  - players' weights.
  - the distance cost of being disconnected;

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# Basics (graph)

- $G = (V, E)$: an undirected graph.
- $d_G(u, v)$: the *distance* between $u$ and $v$ in $G$.
- $\text{diam}(u) = \max_{v \in V} d_G(u, v)$.
  - $\text{diam}(G) = \max_{v \in V} \text{diam}(v)$.
- For $U \subseteq V$, $w(U) = \sum_{u \in U} x_u$.
  - ⋆ $W = w(V)$.
- $w_{\max} = \max_{u \in V} w_u$.
- $w_{\min} = \min_{u \in V} w_u$.

# Basics (The model)

## Celebrity Game

- A celebrity game $\Gamma$ is a tuple $\langle V, (w_u)_{u \in V}, \alpha, \beta \rangle$ where:
  - $V = \{1, \ldots, n\}$ is the set of $n$ players;
  - $w_u > 0$ for each $u \in V$ is the weight of player $u$;
  - $\alpha > 0$ is the cost of establishing a link;
  - $\beta \in [1, n-1]$: the *critical distance*.

- A strategy for player $u$: $S_u \subseteq V \setminus \{u\}$.
  - $\mathcal{S}(u)$: the set of strategies for $u$.
- A strategy profile: $S = (S_1, \ldots, S_n)$.
  - $\mathcal{S}(\Gamma)$: the set of strategy profiles of $\Gamma$.
- Every strategy profile associates an outcome graph $G[S] = (V, \{\{u, v\} \mid u \in S_v \vee v \in S_u\})$.

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# Basics (cost & NE)

- $c_u(S) = \alpha|S_u| + \sum_{\{v|d_{G[S]}(u,v)>\beta\}} w_v$.
  - $C(S) = \sum_{u \in V} c_u(S)$.
- The cost difference $\Delta(S_{-u}, S'_u) := c_u(S_{-u}, S'_u) - c_u(S)$.
  - $(S_{-u}, S'_u)$: $S_u$ is replaced by $S'_u$, the others remain unchanged.
- Best response to $S \in \mathcal{S}(\Gamma)$ for a player $u$:
  a strategy $S'_u \in \mathcal{S}(u)$ minimizing $\Delta(S_{-u}, S'_u)$.

**Nash equilibrium (NE)**

Let $\Gamma = \langle V, (w_u)_{u \in V}, \alpha, \beta \rangle$ be a celebrity game. A strategy profile $S \in \mathcal{S}(\Gamma)$ is a Nash equilibrium of $\Gamma$ if

$$\forall u \in V, \forall S'_u \in \mathcal{S}(u), \text{ such that } \Delta(S_{-u}, S'_u) \geq 0.$$

# Basics (More on total cost)

- For $G = G[S]$,

$$C(G) = \alpha|E(G)| + \sum_{u \in V} \sum_{\{v|\ d_G(u,v) > \beta\}} w_v$$
$$= \alpha|E(G)| + \sum_{\{(u,v)|\ u<v \wedge d_G(u,v)>\beta\}} (w_v + w_u).$$

- Particular cases:
  - $C(K_n) = \alpha n(n-1)/2.$
  - $C(I_n) = W(n-1)$ for $\beta \geq 1$.
  - $C(ST_n) = \alpha(n-1)$ for $1 < \beta \leq n-1$, and $C(ST_n) = \alpha(n-1) + (n-2)(W - w_c)$, where $c$ is the central vertex, for $\beta = 1$.

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# Basics (PoS & PoA)

- $\text{PoA}(\Gamma) = \max_{S \in NE(\Gamma)} C(S)/\text{opt}(\Gamma)$.
- $\text{PoS}(\Gamma) = \min_{S \in NE(\Gamma)} C(S)/\text{opt}(\Gamma)$.

# NP-hard for computing a best response

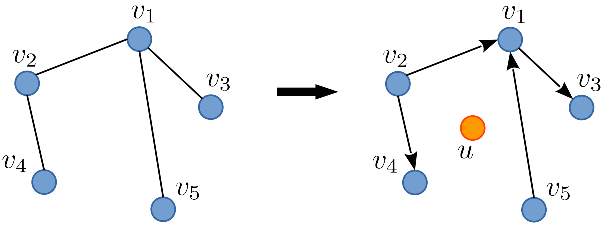
**Proposition 1**

Computing a best response for a player to a strategy profile in a celebrity game is **NP**-hard.

- Reduction from minimum dominating set.

# Proof of Proposition 1

$G = (V, E)$




$V' = V \cup \{u\}$

$\beta = 2,\ \alpha = 1.5$

$\forall v \in V,\ w_v = 2$

$S_{v_1} = \{v_2, v_5\}$

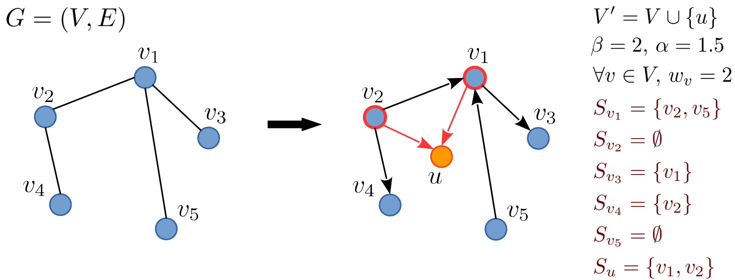
$S_{v_2} = \emptyset$

$S_{v_3} = \{v_1\}$

$S_{v_4} = \{v_2\}$

$S_{v_5} = \emptyset$

# Proof of Proposition 1 (contd.)




- Let $D \subseteq V$ be a strategy of $u$.
- If $D$ is a dominating set of $G$:
  - $c_u(S_{-u}, D) = \alpha|D| + \sum_{x \in V, d(u,x)>2} 2 = \alpha|D|$.
- Otherwise,

$$c_u(S_{-u}, D) = \alpha|D| + \sum_{x \in V, d(u,x)>2} 2 > \alpha(|D| + |\{x \in V \mid d(u,x) > 2\}|).$$

# Social Optimum and Nash Equilibria

## Proposition 2

For a celebrity game $\Gamma = \langle V, (w_u)_{u \in V}, \alpha, \beta \rangle$, $\mathrm{opt}(\Gamma) = \min\{\alpha, W\}(n-1)$.

- Let $S \in OPT(\Gamma)$ and $G = G[S]$ with connected components $G_1, \ldots, G_r$.
    - $V_i = V(G_i)$, $k_i = |V_i|$, and $W_i = w(V_i)$, for each $i$.
- Each $G_i$ must be a tree of diameter $\leq \beta$.
- $$C(G) = \sum_{i=1}^{r} \alpha(k_i - 1) + \sum_{i=1}^{r} k_i(W - W_i) = \alpha(n-r) + nW - \sum_{i=1}^{r} k_i W_i.$$
- $\because 1 \leq k_i \leq n-(r-1) \quad \therefore W \leq \sum_{i=1}^{r} k_i W_i \leq (n-r+1)W$.

  $\triangleright \ \alpha(n-r) + (r-1)W \leq C(G)$.

Consider the two cases below.

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Consider the two cases below.

# Proof of Proposition 2 (contd.)

Recall that: $\alpha(n-r)+(r-1)W \le C(G)$

- Case I: $\alpha \ge W$.
  - $W(n-1) \le \alpha(n-r)+(r-1)W \le C(G)$.
  - Since $C(I_n) = W(n-1)$ and $G$ is optimal, then $C(G) = W(n-1)$.
- Case II: $\alpha < W$.
  - $\alpha(n-1) \le C(G)$.
  - Since $C(ST_n) = \alpha(n-1) \le C(G)$, then $C(G) = \alpha(n-1)$.

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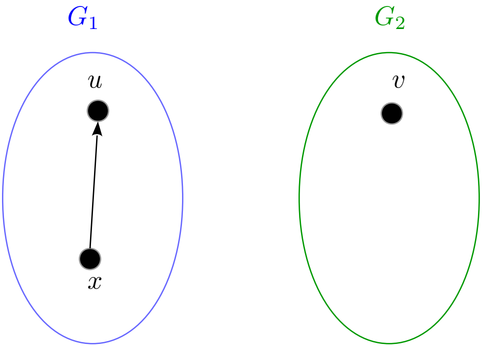
## Proposition 3

Every NE graph of a celebrity game $\Gamma = \langle V, (w_u)_{u \in V}, \alpha, \beta \rangle$ is either connected or the graph $I_n$.



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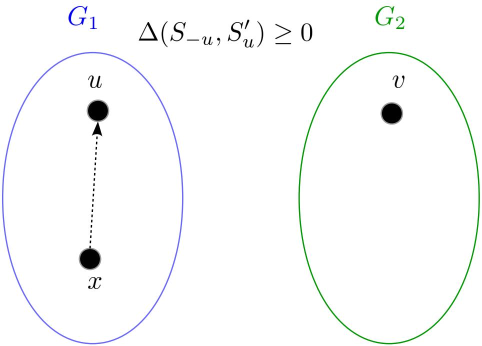

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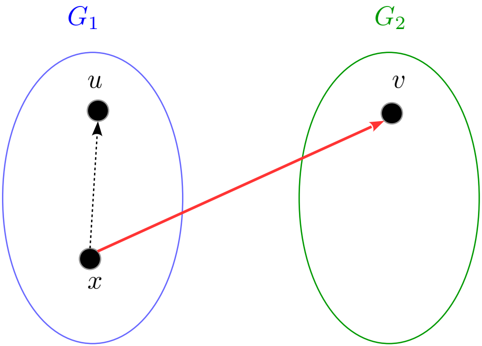


$$S'_u = S_u \setminus \{x\},\ S'_v = S_v = S_v \cup \{x\}$$

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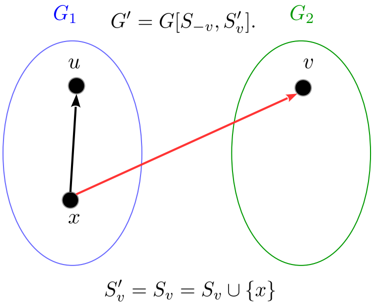


- $d_{G'}(v, u) = 2 \leq \beta \quad \Rightarrow$
  $\Delta(S_{-v}, S'_v) \leq -\Delta(S_{-u}, S'_u) - w_u < 0$
  ($\Rightarrow\Leftarrow$ $S$ is a NE).

## Proposition 4

- Every celebrity game $\Gamma = \langle V, (w_u)_{u \in V}, \alpha, \beta \rangle$ has a NE.
- $\alpha \geq w_{\max}$: $I_n$ is a NE graph;
  otherwise: $ST_n$ is a NE graph but $I_n$ is NOT.

- $\alpha \geq w_{\max}$:
  - Let $G = G[S] = I_n$.
    - $S_u = \emptyset, \forall u \in V$.
    - Consider $u \in V$ and $S'_u \neq \emptyset$.
    - $\Delta(S_{-u}, S'_u) = \alpha |S'_u| - \sum_{v \in S'_u} w_v = \sum_{v \in S'_u} (\alpha - w_v) \geq 0$.

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- $\alpha < w_{\max}$:
  - Let $u$ be a vertex with $w(u) = w_{\max}$.
  - Let $ST_n$ be a star graph with $u$ being the center.
    - $S_u = \emptyset$, and $S_v = \{u\}$, $\forall v \in V \setminus \{u\}$.
    - $\beta > 1 \Rightarrow$ NO player gets a cost decrease by connecting to more players.
    - For $u \neq v$, $w_v + \alpha < w_v + w_{\max} < W \Rightarrow \alpha < W - w_v$ (deleting any connection won't help).
    - $I_n$ can NOT be a NE graph.
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  otherwise: $ST_n$ is a NE graph but $I_n$ is NOT.

- $\alpha < w_{\max}$:
  - Let $u$ be a vertex with $w(u) = w_{\max}$.
  - Let $ST_n$ be a star graph with $u$ being the center.
    - $S_u = \emptyset$, and $S_v = \{u\}$, $\forall v \in V \setminus \{u\}$.
    - $\beta > 1 \Rightarrow$ NO player gets a cost decrease by connecting to more players.
    - For $u \neq v$, $w_v + \alpha < w_v + w_{\max} < W \Rightarrow \alpha < W - w_v$ (deleting any connection won't help).
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## Proposition 5

Let Γ be a celebrity game with $\alpha \geq w_{\max}$.

- If there is MORE THAN ONE vertex $u \in V$ with $\alpha > W - w_u$, then $I_n$ is the UNIQUE NE graph of Γ;
- otherwise, $ST_n$ is a NE graph of Γ.

## Corollary 1

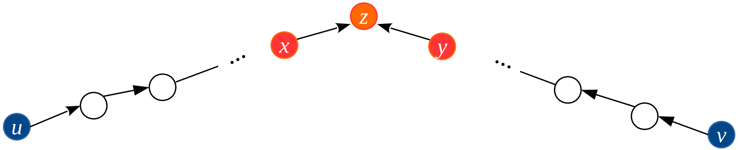
Let Γ be a celebrity game.

$I_n$ is the unique NE graph of Γ if and only if

- $\alpha \geq w_{\max}$ and
- there is more than one vertex $u \in V$ s.t. $\alpha > W - w_u$.

# Proof of Proposition 5

- Assume that $\exists u, v \in V$, $\alpha > W - w_u$ and $\alpha > W - w_v$, and there is a NE graph $G = G[S] \neq I_n$.
- $G$ is connected (by Proposition 3).
- $S_u = S_v = \emptyset$ ($\because \alpha > W - w_u$ and $\alpha > W - w_v$).




$|S_z| \geq 2, \quad x, y \in S_z$

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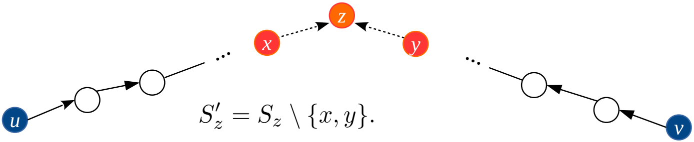


$$S'_z = S_z \setminus \{x, y\}.$$

$$\Delta(S_{-z}, S'_z) \leq -2\alpha + W - w_z.$$

$$\because G \text{ is a NE graph and } 2\alpha > W - w_u + W - w_v$$

$$\therefore W - w_z \geq 2\alpha > W - W_u + W - w_v.$$

$$\Rightarrow W < w_u + w_v - w_z < w_u + w_v \text{ (impossible)}$$

# Proof of Proposition 5 (contd.)

- Case II: at most one vertex $u$ with $\alpha > W - w_u$.
  - $S$: $S_u = \emptyset$, and $S_v = \{u\}$, $\forall v \neq u$.
  - $S$ is NE & $G[S] = ST_n$.

## Star Celebrity Game

$\Gamma$ is a star celebrity game if $\Gamma$ has a NE graph that is connected.

## Corollary 2

For a celebrity game $\Gamma$, the following statements are equivalent.

- $\Gamma$ is a star celebrity game;
- Either $\alpha < w_{\max}$ or $\alpha \geq w_{\max}$ and there is at most one vertex $u \in V$ for which $\alpha > W - w_u$.
- $ST_n$ is a NE graph of $\Gamma$.

**Theorem 1**

Let Γ be a celebrity game. Then we have

- If Γ is a star celebrity game, $\mathrm{PoS}(\Gamma) = 1$.
- If Γ is NOT a star celebrity game and $\alpha \geq W$, then $\mathrm{PoS}(\Gamma) = \mathrm{PoA}(\Gamma) = 1$.
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- $\mathrm{opt}(\Gamma) = W(n-1)$ if $\alpha \geq W$;
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# Bounding the PoA of Star Celebrity Games

# Bounding the PoA

## Lemma 2

For a star celebrity game $\Gamma$, $\mathrm{PoA}(\Gamma) \leq W/\alpha$.

- Let $S$ be a NE of $\Gamma$ and let $G = G[S] = (V, E)$.
- $0 \leq \Delta(S_{-u}, \emptyset) \leq -\alpha|S_u| + w(\{v \mid d(u,v) \leq \beta\}) - w_u$.
- Thus, for all $u \in V$,

$$0 \leq \sum_{u \in V}(-\alpha|S_u| + \sum_{\{v \mid d(u,v) \leq \beta\}} w_v - w_u) = -\alpha|E| + \sum_{u \in V} \sum_{\{v \mid d(u,v) \leq \beta\}} w_v - W.$$

- Therefore,

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# Bounding the PoA (contd.)

## Weight component

$$W(G,\beta) = \sum_{u\in V} \sum_{\{v|d(u,v)>\beta\}} w_v = \sum_{\{\{u,v\}|d(u,v)>\beta\}} (w_u + w_v).$$

## Lemma 3

Let $\Gamma$ be a star celebrity game. In a NE graph $G$,

$$W(G,\beta) = O(\alpha n^2/\beta).$$

- Let $S$ be a NE and $G = G[S]$ be a NE graph.
- Let $b = \text{diam}(u)$ for $u \in V$.
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- Consider the following three cases.

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Let $\Gamma$ be a star celebrity game. In a NE graph $G$,

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- Let $S$ be a NE and $G = G[S]$ be a NE graph.
- Let $b = \text{diam}(u)$ for $u \in V$.
  - $b \leq 2\beta + 1$ (by Proposition 6; we prove it later).
- Consider the following three cases.

# Proof of Lemma 3

- Case 1: $b < \beta$:
  - For any $v \in V \setminus \{u\}$, consider $S'_v = S_v \cup \{u\}$ and let $G' = G[S_{-v}, S'_v]$.
  - $\text{diam}(G') \leq \beta$.
  - $\Delta(S_{-v}, S'_v) = \alpha - \sum_{\{x \mid d_G(x,v) > \beta\}} w_x \geq 0$.

$$\therefore \sum_{\{x \mid d_G(x,v) > \beta\}} w_x \leq \alpha.$$

  - Hence, $W(G, \beta) \leq n\alpha \leq \alpha n^2/\beta$ (note: $1 < \beta \leq n-1$).

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# Proof of Lemma 3 (contd.)

- Case 2: $b \geq \beta$ and $b \geq 6$:
  - $A_i(u) := \{v \mid d(u,v) = i\}$, and

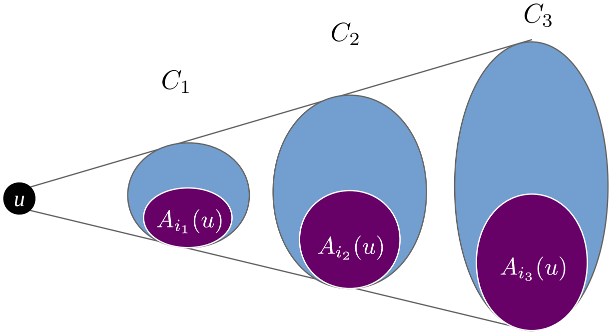
$$C_1 = \{v \in V \mid 1 \leq d(u,v) \leq b/3\} = \bigcup_{1 \leq i \leq b/3} A_i(u),$$

$$C_2 = \{v \in V \mid b/3 \leq d(u,v) \leq 2b/3\} = \bigcup_{b/3 \leq i \leq 2b/3} A_i(u),$$

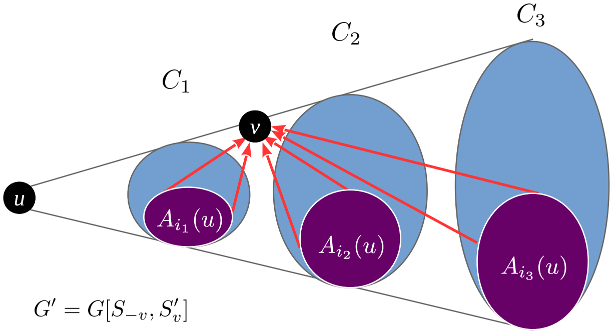
$$C_3 = \{v \in V \mid 2b/3 \leq d(u,v) \leq b\} = \bigcup_{2b/3 \leq i \leq b} A_i(u).$$

  - Each $C_\ell$ contains vertices at $b/3 \geq 2$ different distances.
  - For each $1 \leq \ell \leq 3$, there must exist $i_\ell$ s.t. $A_{i_\ell} \subseteq C_\ell$ and $|A_{i_\ell}| \leq 3n/b$.

## Proof of Lemma 3 (Case 2: $b \geq \beta$ and $b \geq 6$)



## Proof of Lemma 3 (Case 2: $b \geq \beta$ and $b \geq 6$)




$$G' = G[S_{-v}, S'_v]$$

$$\forall v \in V, \quad S'_v = (S_v \cup A_{i_1}(u) \cup A_{i_2}(u) \cup A_{i_3}(u)) \setminus \{v\}$$

- Note: $b/3 < \beta$ ($\because b \leq 2\beta + 1$).
  - $\mathrm{diam}_{G'}(v) \leq \beta$.
- $0 \leq \Delta(S_{-v}, S'_v) \leq \frac{9n\alpha}{\beta} - \sum_{\{x \mid d_G(x,v) > \beta\}} w_x \quad \Rightarrow \quad W(G,\beta) \leq \frac{9n^2\alpha}{\beta}$.

# Proof of Lemma 3 (contd.)

- Case 3: $b \geq \beta$ and $b \leq 6$:
    - Similar to case 2.

# Bounding the PoA (contd.)

## Lemma 4

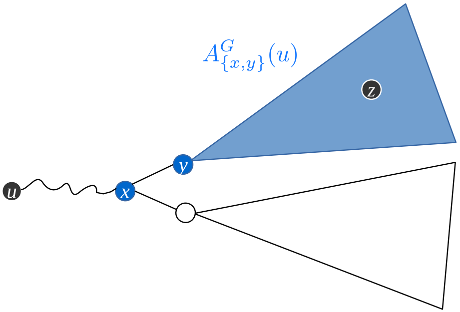
Let Γ be a star celebrity game. In a NE graph $G$,

$$|E(G)| \leq n - 1 + \frac{3n^2}{\beta}.$$

# Critical nodes

## Critical nodes

$A^G_{\{x,y\}}(u) = \{z \in V \mid$ **all** the shortest paths in $G$ from $u$ to $z$ use $\{x,y\}\}$.



# Proof of Lemma 4 (contd.)

**Bridge (cut edge)**

An edge whose deletion increases # connected components of the graph.

- $B(G)$: the set of bridges of $G$.
  - $|B(G)| \leq n - 1$.
- $\bar{B}(u) = \{x \in S_u \mid \{u, x\} \notin B(G)\}$.
- $|E| = |B(G) + \sum_{u \in V} |\bar{B}(u)|$
- Claim: for any $v \in S_u$ s.t. $\{u, v\}$ is NOT a bridge, there exists $z \in A_{\{u,v\}}(u)$ s.t. $d(u, z) > \beta/3$.

# Proof of Lemma 4 (contd.)

**Bridge (cut edge)**

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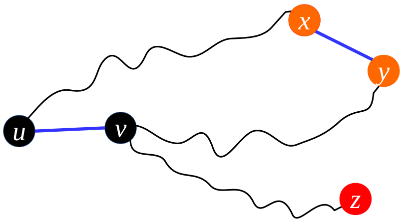
# Proof of Lemma 4 (contd.)

$\{u, v\}$ : NOT a bridge
$z \in A_{\{u,v\}}(u), \;\; d(u, z) \leq \beta/3$



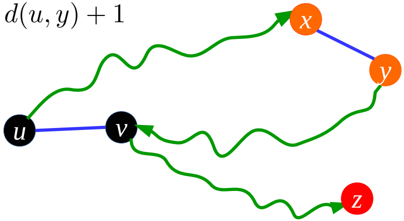

There must exist some edge $\{x, y\}$, such that

$$x \notin A_{\{u,v\}}(u), \; y \in A_{\{u,v\}}(u)$$

Select $x \neq u$ s.t. there is a shortest path from $u$ to $x$ using only vertices in $V \setminus A_{\{u,v\}}(u)$.

# Proof of Lemma 4 (contd.)

$d(u,x) \leq d(u,y) + 1$




The new path (not using $\{u,v\}$) from $u$ to $z$ is of distance $\leq (\beta/3+1)+1+(\beta/3-1)+(\beta/3-1)=\beta$.

$\Delta(S_{-u}, S_u \setminus \{v\}) = -\alpha < 0 \ (\Rightarrow\!\Leftarrow)$

# Proof of Lemma 4 (contd.)

- For $v \in \bar{B}(u)$, there exists $z \in A_{\{u,v\}}(u)$ s.t. $d(u,z) > \beta/3$.
  $\therefore\ |A_{\{u,v\}}(u)| > \beta/3$.
- $n \geq \sum_{\{v \in S_u | v \in \bar{B}(u)\}} |A_{u,v}(u)| \geq |\bar{B}(u)| \cdot (\beta/3)$.
  $\therefore\ |\bar{B}(u)| \leq \frac{3n}{\beta}$.
- $|E| = |B(G)| + \sum_{u \in V} |\bar{B}(u)| \leq (n-1) + \frac{3n^2}{\beta}$.

# Bounding the PoA (contd.)

**Theorem 2**

For a star celebrity game $\Gamma$, $\mathrm{PoA}(\Gamma) = O(\min\{n/\beta, W/\alpha\})$.

- $C(G) \leq \alpha \cdot |E| + W(G, \beta) = O\left(\alpha \cdot \left((n-1) + \frac{3n^2}{\beta}\right) + \frac{\alpha n^2}{\beta}\right)$.

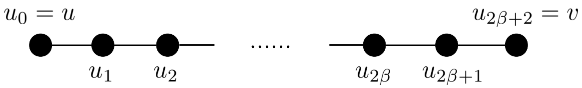
$\therefore \frac{C(G)}{\alpha(n-1)} = O\left(\frac{n}{\beta}\right)$.

# Diameter of a NE graph of Γ

**Proposition 6**

If $G$ is a NE graph of a star celebrity game Γ, then $\text{diam}(G) \leq 2\beta + 1$.

- Let $S$ be a NE of Γ, $G = G[S]$, and assume that $\text{diam}(G) \geq 2\beta + 2$.
- $\exists u, v \in V$ s.t. $d(u, v) = 2\beta + 2$.

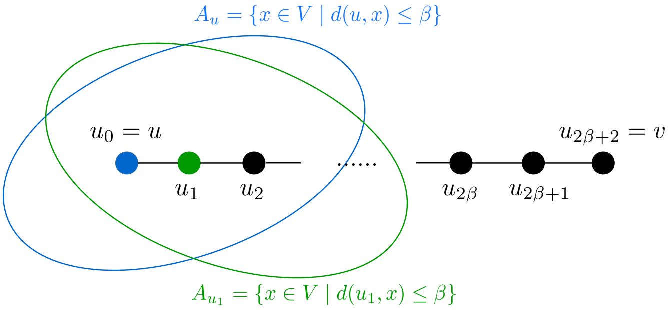


# Diameter of a NE graph of Γ

**Proposition 6**

If $G$ is a NE graph of a star celebrity game Γ, then $\text{diam}(G) \leq 2\beta + 1$.

- If $x \in A_u \cup A_{u_1}$, then $d(x, v) > \beta$.



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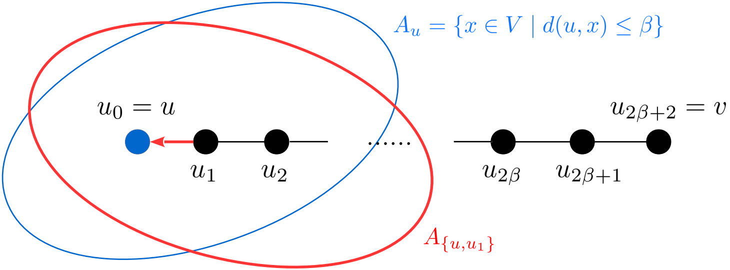


Consider the case $u_1 \in S_u$

# Diameter of a NE graph of Γ

## Proposition 6

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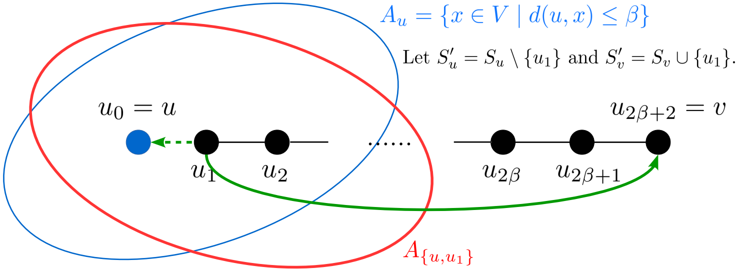


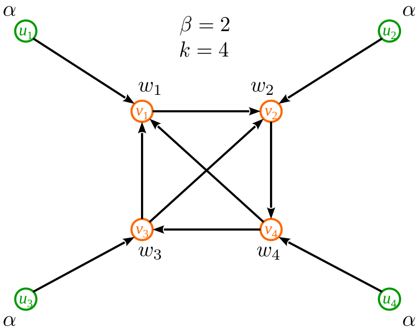

$$\left.\begin{array}{l}\Delta(S_{-u}, S'_u) \leq -\alpha + w(A_{\{u,u_1\}} \cap A_u). \\ \Delta(S_{-v}, S'_v) \leq \alpha - w_u - w(A_{\{u,u_1\}} \cap A_u).\end{array}\right\} \Rightarrow \Delta(S_{-u}, S'_u) + \Delta(S_{-v}, S'_v) \leq -w_u < 0.$$

either $\Delta(S_{-u}, S'_u) < 0$ or $\Delta(S_{-v}, S'_v) < 0$. $(\Rightarrow\Leftarrow)$

# Bounding the PoA (a lower bound)

**Lemma 5**

Let $k > 2$, $\alpha > 0$, and let $w = (w_1, \ldots, w_k)$ be a positive weight assignment. There is a start celebrity game $\Gamma$ with $n = 2k$ and $\beta = 2$, s.t. $\text{PoA}(\Gamma) > \frac{3n}{8}$.




$$\text{PoA}(\Gamma) = \frac{\alpha\left(\frac{k(k-1)}{2} + k\right) + \alpha k(k-1)}{\alpha(n-1)} = \frac{3n}{8}\left(1 + \frac{1}{3(n-1)}\right).$$

# PoA on NE trees

**Theorem 5**

The PoA on NE trees of star celebrity games is $\leq 5/3$, and there are games for which a NE tree has cost $5 \cdot \mathrm{opt}/3$.

# Celebrity Games for $\beta = 1$

# For $\beta = 1$

- Every $u \in V$ pays
  - $w_v$ for each non-adjacent $v \in V \setminus \{u\}$.
  - $\alpha$ for each adjacent $v \in V \setminus \{u\}$ if he buys the link.

**Proposition 10**

The problem of computing a best response of a player to a strategy profile in celebrity games is polynomial time solvable when $\beta = 1$.

**Theorem 6**

Let $\Gamma$ be a celebrity game with $\beta = 1$. We have $\mathrm{PoA}(\Gamma) \leq 2$.

# For $\beta = 1$ (PoA)

## Proposition 11

Let $G = (V, E)$ be an NE graph of a celebrity game $\Gamma$ with $\beta = 1$. For each $u, v \in V$, we have

- if either $w_u > \alpha$ or $w_v > \alpha$, then $\{u, v\} \in E$,
- if both $w_u < \alpha$ and $w_v < \alpha$, then $\{u, v\} \notin E$,
- otherwise $\{u, v\}$ might or might not belong to $E$.

## Proposition 12

Let $G = (V, E)$ be an OPT graph of a celebrity game $\Gamma$ with $\beta = 1$. For any $u, v \in V$, we have

- if $w_u + w_v < \alpha$, then $\{u, v\} \notin E$,
- if $w_u + w_v > \alpha$, then $\{u, v\} \in E$,
- if $w_u + w_v = \alpha$, then $\{u, v\}$ might or might not belong to $E$.

# Proof of Theorem 6

- The social cost of an OPT graph:

$$\sum_{\{\{u,v\} \mid w_u+w_v \geq \alpha\}} \alpha + \sum_{\{\{u,v\} \mid w_u+w_v < \alpha\}} (w_u + w_v).$$

- The social cost of a NE graph (with fewest edges):

$$\begin{aligned} & \sum_{\{\{u,v\} \mid w_u > \alpha \text{ or } w_v > \alpha\}} \alpha + \sum_{\{\{u,v\} \mid w_u, w_v \leq \alpha\}} (w_u + w_v) \\ = & \sum_{\{\{u,v\} \mid w_u > \alpha \text{ or } w_v > \alpha\}} \alpha + \sum_{\{\{u,v\} \mid w_u, w_v \leq \alpha \text{ and } w_u+w_v = \alpha\}} \alpha \\ + & \sum_{\{\{u,v\} \mid w_u, w_v \leq \alpha \text{ and } w_u+w_v < \alpha\}} (w_u + w_v) + \sum_{\{\{u,v\} \mid w_u, w_v \leq \alpha \text{ and } w_u+w_v > \alpha\}} (w_u + w_v). \end{aligned}$$

# Proof of Theorem 6

- The social cost of an OPT graph:

$$\sum_{\{\{u,v\}|w_u+w_v \geq \alpha\}} \alpha + \sum_{\{\{u,v\}|w_u+w_v < \alpha\}} (w_u + w_v).$$

- The social cost of a NE graph (with fewest edges):

$$\begin{aligned}
& \sum_{\{\{u,v\}|w_u>\alpha \text{ or } w_v>\alpha\}} \alpha + \sum_{\{\{u,v\}|w_u,w_v \leq \alpha\}} (w_u + w_v) \\
= & \sum_{\{\{u,v\}|w_u>\alpha \text{ or } w_v>\alpha\}} \alpha + \sum_{\{\{u,v\}|w_u,w_v \leq \alpha \text{ and } w_u+w_v=\alpha\}} \alpha \\
+ & \sum_{\{\{u,v\}|w_u,w_v \leq \alpha \text{ and } w_u+w_v<\alpha\}} (w_u + w_v) + \sum_{\{\{u,v\}|w_u,w_v \leq \alpha \text{ and } w_u+w_v>\alpha\}} (w_u + w_v).
\end{aligned}$$

# Proof of Theorem 6 (contd.)

- $D := \{\{u, v\} \mid w_u, w_v \leq \alpha \text{ and } w_u + w_v > \alpha\}$.
- $\{u, v\}$ contributes:
  - $\alpha$ to the cost of an OPT graph;
  - $w_u + w_v$ to the cost of a NE graph.
- Taking $w_u = \alpha$ for any $u \in V$ to maximize $|D|$.

$$\text{PoA}(\Gamma) \leq \frac{n(n-1)\alpha}{\alpha n(n-1)/2} = 2.$$

# Open problems

- Shorten the gap b/w LB and UB on the PoA of the celebrity games (for constant $\beta$).
- Variations of the framework:
  - Max-cost model (authors' work in progress).
  - Other definitions of the social cost.
  - Each player $u$ can have its own critical distance $\beta_u$.

Thank you.