On Discrete Preference and Coordination

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How Bad is Forming Your Own Opinion?

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strategy
node
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Nash
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response
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PoS
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component
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statement
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minimize
coordinate
reduce
available
space
question
family
path
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tight
central
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Sexes
payoff
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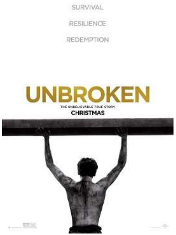
Outline

Coordination with discrete preferences

- A classic example:
  - Battle of the Sexes.
    - Joseph wants to see “Unbroken”.
    - Maggie wants to see “Gone Girl”.

- Characteristics:
  1. conflicting internal preferences;
  2. an incentive to arrive at a compromise;
  3. no way to *average* between the options.



Coordination with discrete preferences

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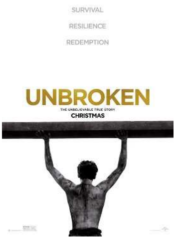


Contribution of this paper

- Develop model and techniques for analyzing *discrete preference games*.
- Price of stability results.
  - PoS $= 1$ when the two effects "network coordination" and 'unilateral decision effects" are balance and a tree metric on the strategy set is used.
  - PoS $\nearrow 2$ for non-tree metrics.

Basic terminology

- L: the strategy set.
- $G = (V, E)$: the undirected graph where the game play is played.
  - V: the set of players.
  - E: the edge set (players' relations on the network).
- $s_i \in L$: the preferred strategy of player $i \in V$.
- $d(\cdot, \cdot)$: a distance metric on L.
  - $d(i, i) = 0$ for all i;
  - $d(i, j) = d(j, i)$ for all i, j;
  - $d(i, j) \leq d(i, k) + d(k, j)$ for all i, j, k.

Player's cost & the social cost

All players choose strategies $z = \langle z_j : j \in V \rangle$; $\alpha \in [0,1]$.

- The cost incurred by player i:

$$c_i(z) = \alpha \cdot d(s_i, z_i) + \sum_{j \in N(i)} (1-\alpha) \cdot d(z_i, z_j).$$

- The social cost of the game:

$$c(z) = \sum_{i \in V} \alpha \cdot d(s_i, z_i) + 2 \sum_{(i,j) \in E} (1-\alpha) \cdot d(z_i, z_j).$$

- The *contribution of player i* to the social cost of the game:

$$sc_i(z) = \alpha \cdot d(s_i, z_i) + 2 \sum_{j \in N(i)} (1-\alpha) \cdot d(z_i, z_j).$$

Why not PoA (price of anarchy)?

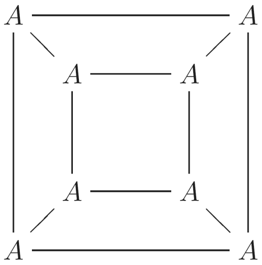
PoA could be unbounded ($0 < \alpha < 1$)

Assume that $L = \{A, B\}$ and $d(A, B) = 1$.

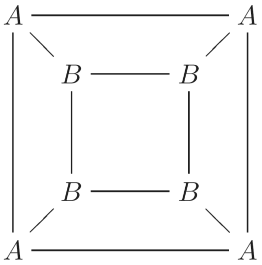
- Consider a clique of size $\lceil \frac{\alpha}{1-\alpha} \rceil + 1$.
  - All the players prefer A.
- An equilibrium: all the players play B.
  - The cost of player i for playing A: $\alpha \cdot 0 + (1-\alpha) \cdot \lceil \frac{\alpha}{1-\alpha} \rceil \geq \alpha$.
- Optimal solution: every player plays A (cost: 0).

PoA could be unbounded ($\alpha = 0$)

Network coordination games.

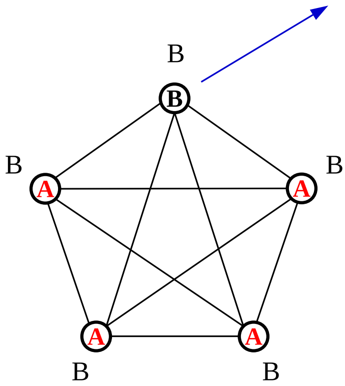


(a) *An optimal solution* .



(b) *A Nash equilibrium* .

PoA could be unbounded even for strong NE



No gain from any deviation

If: "the rest players simultaneously deviate to A"

For any of these players:

cost before: $\alpha < 1/2$
cost after: $1-\alpha$

the cost increase by
$(1-\alpha) - \alpha > 0$

Cost of this best Nash equilibrium: $(n-1)\alpha$
Cost of the optimal solution: α

The price of stability (PoS) is bounded by 2

$$\phi(z) = \alpha \sum_{i \in V} d(z_i, s_i) + (1-\alpha) \sum_{(i,j) \in E} d(z_i, z_j).$$

- $\phi(\cdot)$ is an exact potential function.

$$\begin{aligned}
& \phi(z_i, z_{-i}) - \phi(z_i', z_{-i}) \\
= \; & \alpha \cdot d(z_i, s_i) + (1-\alpha) \sum_{j \in N(i)} d(z_i, z_j) - \left(\alpha \cdot d(z_i', s_i) + (1-\alpha) \sum_{j \in N(i)} d(z_i', z_j) \right) \\
= \; & c_i(z_i, z_{-i}) - c_i(z_i', z_{-i}).
\end{aligned}$$

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The price of stability (PoS) is bounded by 2 (contd.)

- x: the global minimizer of $\phi(\cdot)$.
  - x is a Nash equilibrium ($\because$ ϕ is a potential function).
- y: the optimal solution.

$$c(x) \leq 2\phi(x) \leq 2\phi(y) \leq 2c(y).$$

The case of two strategies

The main results for $|L| = 2$

Claim 3.4

If $\alpha \leq \frac{1}{2}$ or $\alpha = \frac{2}{3}$, then in any instance there exists an optimal solution which is also a Nash equilibrium.

Theorem 3.5

For $\frac{1}{2} < \alpha < 1$, PoS $\leq 2\lceil \frac{\alpha}{1-\alpha} - 1 \rceil \cdot \frac{1-\alpha}{\alpha}$.

Claim 3.7

For any $1 > \alpha > 1/2$, $\alpha \neq \frac{2}{3}$, there exists an instance achieving a PoS of $2\lceil \frac{\alpha}{1-\alpha} - 1 \rceil \cdot \frac{1-\alpha}{\alpha}$.

Proof of Claim 3.4

Claim 3.4

If $\alpha \leq \frac{1}{2}$ or $\alpha = \frac{2}{3}$, then in any instance there exists an optimal solution which is also a Nash equilibrium.

- Let y be an optimal solution minimizing $\phi(\cdot)$.
- Assume that it is NOT a Nash equilibrium.
- Player i prefers to switch to a best response x_i.
- We derive $y_i \neq s_i$ and $x_i = s_i$ (by Observations 3.1 & 3.2).
  - If $y_i = s_i$, then the strategy minimizing player i's cost is also s_i.

Two observations

- $L = \{A, B\}$, $d(A, B) = 1$.
- $N_j(i)$: player i's neighbors using strategy j.
- $\bar{s}_i$: the strategy opposite to s_i.

Observation 3.1

The strategy s_i minimizes player i's cost (i.e., $c_i(z)$) if

$$(1-\alpha)N_{\bar{s}_i}(i) \leq \alpha + (1-\alpha)N_{s_i}(i)$$

That is, $N_{\bar{s}_i}(i) \leq \frac{\alpha}{1-\alpha} + N_{s_i}(i)$.

Observation 3.2

The strategy s_i minimizes the social cost $sc_i(z)$ if

$$2(1-\alpha)N_{\bar{s}_i}(i) \leq \alpha + 2(1-\alpha)N_{s_i}(i)$$

That is, $N_{\bar{s}_i}(i) \leq \frac{\alpha}{2(1-\alpha)} + N_{s_i}(i)$.

Proof of Claim 3.4 (contd.)

- $y_i \neq s_i$ and $x_i = s_i$.

$$\therefore \quad N_{\bar{s}_i}(i) \lneq \frac{\alpha}{1-\alpha} + N_{s_i}(i).$$

- If s_i minimizes the social cost, then (s_i, y_{-i}) is also an optimal solution.
  - $\phi(s_i, y_{-i}) < \phi(y)$. $(\Rightarrow\Leftarrow)$

$$\therefore \quad \frac{\alpha}{2(1-\alpha)} + N_{s_i}(i) < N_{\bar{s}_i}(i).$$

- Solving $\frac{\alpha}{2(1-\alpha)} < k < \frac{\alpha}{1-\alpha}$ for integer k $\Rightarrow$ $\frac{1}{2} < \alpha < \frac{2}{3}$ or $\alpha > \frac{2}{3}$.

Proof of Claim 3.4 (contd.)

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Proof of Theorem 3.5

Lemma 3.3

Starting from some initial strategy vector, the following best response order results in a Nash equilibrium.

1. While there exists a player that can reduce its cost by changing its strategy to A, let it do the best response.
   - If there is no such player, continue to step 2.
2. While there exists a player that can reduce its cost by changing its strategy to B, let it do the best response.

- An optimal solution $y \xrightarrow{\text{steps above}}$ an equilibrium x.
  - Assume: Only play the unique best response.

Lemma 3.6

Let player i's unique best response be x_i when the rest play z_{-i}, then:

(i). If $x_i = \bar{s}_i$, then $c(x_i, z_{-i}) - c(s_i, z_{-i}) \leq \alpha - 2(1-\alpha)\lfloor \frac{\alpha}{1-\alpha} + 1 \rfloor < 0$.

(ii). If $x_i = s_i$, then $c(x_i, z_{-i}) - c(\bar{s}_i, z_{-i}) \leq -\alpha + 2(1-\alpha)\lceil \frac{\alpha}{1-\alpha} - 1 \rceil$.

- (i) + (ii) ≤ 0 (changing back-and-forth $\Rightarrow$ social cost $\searrow$).

⋆ The only nodes i's capable of increasing the social cost: $y_i \neq s_i$.

  - How many of them? $\sum_i d(y_i, s_i)$.

- Thus,

$$\begin{aligned} c(x) &\leq c(y) + \left(-\alpha + 2(1-\alpha)\left\lceil \frac{\alpha}{1-\alpha} - 1 \right\rceil \right) \sum_{i \in V} d(y_i, s_i) \\ &= 2(1-\alpha) \sum_{(i,j) \in E} d(y_i, y_j) + 2(1-\alpha)\left\lceil \frac{\alpha}{1-\alpha} - 1 \right\rceil \sum_{i \in V} d(y_i, s_i). \end{aligned}$$

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Proof of Theorem 3.5 (contd.)

Theorem 3.5

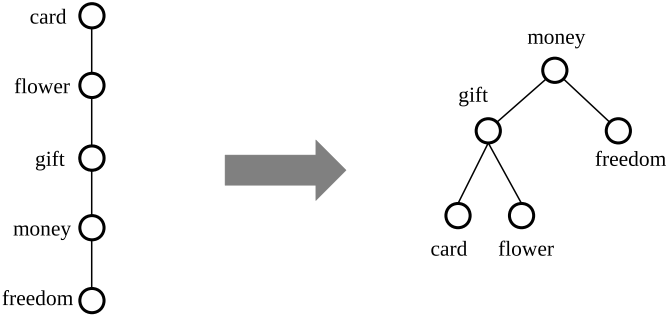
For $\frac{1}{2} < \alpha < 1$, PoS $\leq 2\lceil \frac{\alpha}{1-\alpha} - 1 \rceil \cdot \frac{1-\alpha}{\alpha}$.

$$
\begin{aligned}
\text{PoS} &\leq \frac{2\left\lceil \frac{\alpha}{1-\alpha} - 1 \right\rceil \cdot (1-\alpha)\sum_{i\in V} d(y_i, s_i) + 2(1-\alpha)\sum_{(i,j)\in E} d(y_i, y_j)}{\alpha \sum_{i\in V} d(y_i, s_i) + 2(1-\alpha)\sum_{(i,j)\in E} d(y_i, y_j)} \\
&\leq \frac{2\left\lceil \frac{\alpha}{1-\alpha} - 1 \right\rceil \cdot \frac{1-\alpha}{\alpha} \cdot \left(\alpha \sum_{i\in V} d(y_i, s_i) + 2(1-\alpha)\sum_{(i,j)\in E} d(y_i, y_j)\right)}{\alpha \sum_{i\in V} d(y_i, s_i) + 2(1-\alpha)\sum_{(i,j)\in E} d(y_i, y_j)} \\
&\leq 2\left\lceil \frac{\alpha}{1-\alpha} - 1 \right\rceil \cdot \frac{1-\alpha}{\alpha}.
\end{aligned}
$$

Richer strategy spaces

Tree metrics

- A tree metric (the distance function on the strategy set):
  - the shortest-path among the nodes in a tree.



Strategies for spending the Valentine's day

$C_i(z)$ & $SC_i(z)$ of player i

- $C_i(z)$: the strategies z_i's of player i that minimize

$$c_i(z) = \alpha \cdot d(z_i, s_i) + (1-\alpha) \sum_{j \in N(i)} d(z_i, z_j).$$

- $SC_i(z)$: the strategies z_i's of player i that minimize

$$sc_i(z) = \alpha \cdot d(z_i, s_i) + 2(1-\alpha) \sum_{j \in N(i)} d(z_i, z_j).$$

Claim 4.1

If for every player i and strategy vector z, $C_i(z) \cap SC_i(z) \neq \emptyset$, then PoS $= 1$.

The proof of Claim 4.1

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If for every player i and strategy vector z, $C_i(z) \cap SC_i(z) \neq \emptyset$, then PoS $= 1$.

- Consider y: an optimal solution minimizing $\phi(\cdot)$.
- Assume that y is not a Nash equilibrium.
  - $\exists i \in V$ that can strictly reduce its cost by performing a best response.
- Choose a strategy $x_i \in C_i(z) \cap SC_i(z)$.
  - (x_i, y_{-i}) is also an optimal solution & $\phi(y) > \phi(x_i, y_{-i})$. $(\Rightarrow\Leftarrow)$

The intuition

- a strategy as a player's best response $\leftrightarrow$ a node on the tree not too far away from all the rest nodes from its point of view.
- The concept of medians of a tree.

Definition 4.2

Given a tree T where the weight of node v is denoted by $w(v)$, the set of T's medians is $M(T) = \arg\min_{u \in V} \left\{ \sum_{v \in V} w(v) \cdot d(u, v) \right\}$.

- The detailed proof is based on the following claim:

Claim 4.8

A node u is a median of a tree T iff it is a separator of T.

- A separator of a node-weighted tree T: a node v such that each connected component of $T - v$ is $\leq$ half of the total weight of T.

The proof is omitted here.

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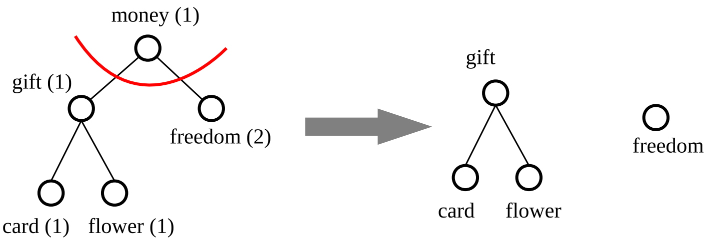
money (1)
gift (1)
freedom (2)
card (1)
flower (1)
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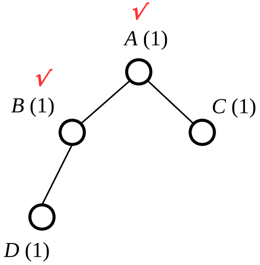

√
A (1)
√
B (1)
C (1)
D (1)

Medians of a node-weighted tree

Definition 4.3

- G: a network,
- T: a tree metric,
- z: a strategy vector,
- i: a player,
- q, r: two non-negative integers,

Denote by $T_{i,z}(q, r)$ the tree T with the following node weights:

$$w(v) = \begin{cases} q + r \cdot |\{j \in N(i) \mid z_j = v\}| & \text{for } v = s_i \\ r \cdot |\{j \in N(i) \mid z_j = v\}| & \text{for } v \neq s_i. \end{cases}$$

The correspondence...

Node weight on $T_{i,z}(q, r)$:

$$w(v) = \begin{cases} q + r \cdot |\{j \in N(i) \mid z_j = v\}|, & \text{for } v = s_i \\ r \cdot |\{j \in N(i) \mid z_j = v\}|, & \text{for } v \neq s_i. \end{cases}$$

Let's see:

$$M(T_{i,z}(a, b)) = \arg\min_{u \in V} \left\{ \sum_{v \in V} w(v) \cdot d(u, v) \right\}$$

$$= \arg\min_{u \in V} \left\{ (a + b|\{j \in N(i) \mid z_j = s_i\}|) \cdot d(u, s_i) + \sum_{v \neq s_i \in V} b|\{j \in N(i) \mid z_j = v\}| \cdot d(u, v) \right\}$$

$$= \arg\min_{u \in V} \left\{ a \cdot d(u, s_i) + b \cdot \sum_{j \in N(i)} d(u, z_j) \right\}$$

$$= C_i(z).$$

- Similarly, $M(T_{i,z}(a, 2b)) = SC_i(z)$.

Proposition 4.4

Let T_1 and T_2 be two node-weighted trees with the same edges and nodes, then:

- If ther exists a node v such that for every $u \neq v \in V$, we have $w_1(u) = w_2(u)$ and for v we have $|w_1(v) - w_2(v)| = 1$, then T_1 and T_2 share a median.
- If T_1 and T_2 share a median, then it is also a median of their union $T_1 \cup T_2$ (i.e., the same nodes and edges yet the weight of v becomes $w_{1+2}(v) := w_1(v) + w_2(v)$).

The proof is omitted here.

Lemma 4.5

For $\alpha = \frac{a}{a+b} \leq \frac{1}{2}$, every player i and strategy vector z,

$$M(T_{i,z}(a,b) \cap M(T_{i,z}(a,2b)) \neq \emptyset.$$

Proof: (mainly by Proposition 4.4)

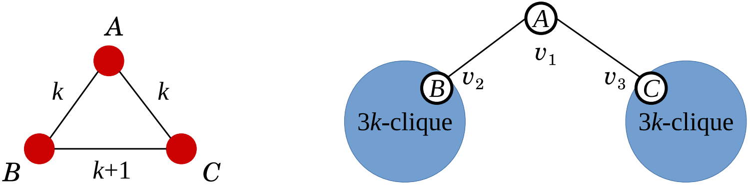
- $T_{i,z}(0,1)$ and $T_{i,z}(1,1)$ share a median u.
- $T_{i,z}(0,b-a)$ and $T_{i,z}(a,a)$ share a median u.
  - Medians are invariant to scaling.
- The median above is also a median of their union $T_{i,z}(a,b)$, and of $T_{i,z}(0,b)$.
  - u is also a median of $T_{i,z}(a,2b)$.

Theorem 4.6 (concluding)

If the distance metric is a tree metric, then for $\alpha \leq \frac{1}{2}$, there exists an optimal solution which is also a Nash equilibrium (i.e., PoS $= 1$).

Lower bounds on the PoS

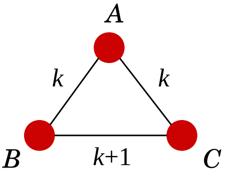
What if the metric is a cycle?



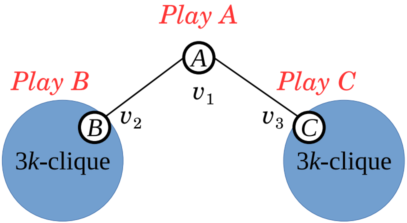
The distance metric

- The best Nash equilibrium has social cost $2k$.
- The optimal solution has social cost $\frac{1}{2} \cdot k + 2 \cdot \frac{1}{2}(k+1) = \frac{3}{2}k + 1$.
- PoS $\nearrow \frac{4}{3}$ as $k \nearrow \infty$.

What if the metric is a cycle?



The distance metric



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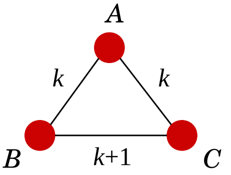
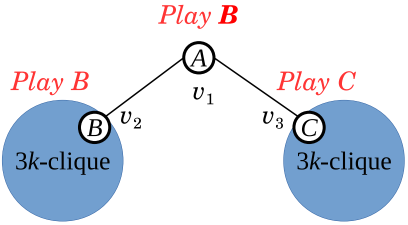


The distance metric



- The best Nash equilibrium has social cost $2k$.
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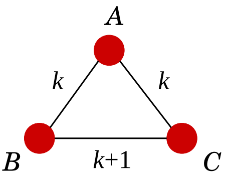
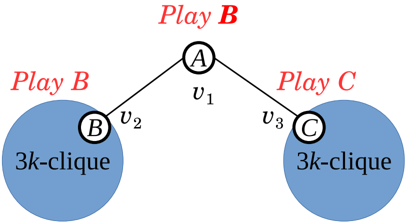
What if the metric is a cycle?

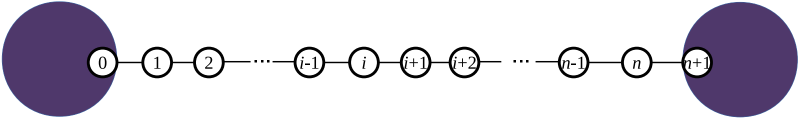


The distance metric



- The best Nash equilibrium has social cost $2k$.
- The optimal solution has social cost $\frac{1}{2} \cdot k + 2 \cdot \frac{1}{2}(k+1) = \frac{3}{2}k + 1$.
- PoS $\nearrow \frac{4}{3}$ as $k \nearrow \infty$.

An example of Pos $\nearrow 2$ $(\alpha = \frac{1}{2})$



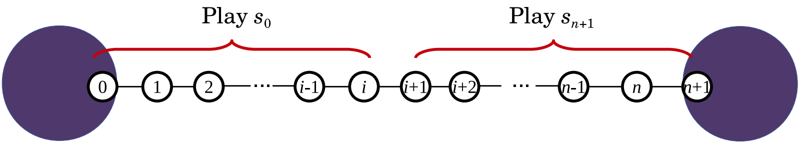
Node i: prefer strategy s_i

● : two cliques of size n^2

$d(s_i, s_j) = 1 + |i - j - 1|\varepsilon$

- The best Nash equilibrium: all players play their preferred strategies.
  - The social cost: $\frac{1}{2} \cdot 2 \sum_{i=0}^{n} d(s_i, s_{i+1}) = n + 1$.

An example of Pos $\nearrow 2$ $(\alpha = \frac{1}{2})$



Node i: prefer strategy s_i

: two cliques of size n^2

$d(s_i, s_j) = 1 + |i - j - 1|\varepsilon$

- The social cost of this assignment: $\frac{1}{2}(n + 2 + O(n^2\epsilon))$.

The Anchored Preference Game

The anchored preference game

- Nodes are partitioned into two types:
  - F: fixed nodes.
    - Always playing their preferred strategy.
  - S: strategic nodes.
    - Having no strategy as preferred.
- The social cost:

$$c(z) = \sum_{\substack{(i,j)\in E;\\ i\in S; j\in F}} d(z_i, s_j) + 2 \sum_{\substack{(i,j)\in E;\\ i,j\in S}} d(z_i, z_j).$$

Generalization of the discrete preference game

- A discrete preference game instance $\rightarrow$ an anchored preference game instance.
  1. For each node i, none of the strategies is preferred.
  2. Add a new *fixed* node i' that has preferred strategy s_i and is connected only to node i by an edge (i, i').
- Discrete preference games are a special case of anchored preference games.
  - *One* fixed neighbor per node.

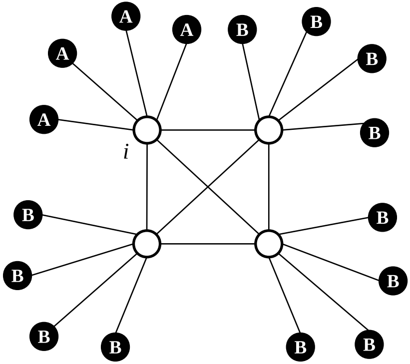
- Consider the parameter k:
  - The maximum number of fixed neighbors of any strategic node.

Claim 6.1

For the anchored preference game, if the distance function is a tree metric, then the following holds.

- If $k \leq 2$, then the optimal solution is also a Nash equilibrium.
- If $k > 2$, then PoS $\leq \frac{2(k-1)}{k}$.

- The bond for $k > 2$ is tight.



- Node i is connected to:
  - k fixed nodes that prefer strategy A.
  - $k-1$ strategic nodes that form a k-clique.
    - Each one is connected to k fixed nodes that prefer strategy B.
- The best NE: node i plays A and the rest of the strategic nodes play B (the social cost: $2(k-1)$).
- Yet, in the optimal solution node i also plays B (the social cost: k).

Thank you.