# Learning-Augmented Facility Location Mechanisms for Envy Ratio

H. AZIZ, Y. Guo, A. Lam, and H. Zhou @NeurIPS 2025

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# Outline



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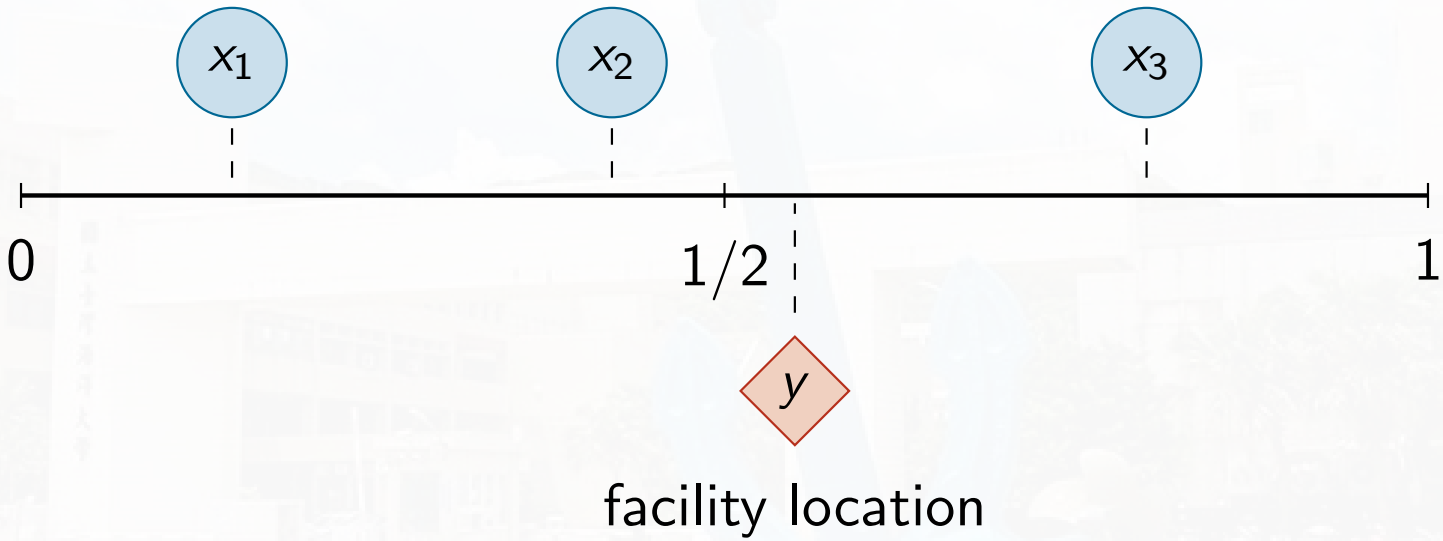
# The Paper in One Sentence

**Main theme**

Can machine-learned predictions help us design strategyproof facility-location mechanisms with better fairness guarantees?

- Setting: one facility on the interval $[0, 1]$.
- Agents want the facility close to their locations.
- Fairness objective: minimize the envy ratio.
  - Firstly proposed by Ding et al. [Comput. Ind. Eng. 2020].
- Prediction: an estimate $\widehat{y}$ of the optimal facility location.

# Classical Facility Location on a Line



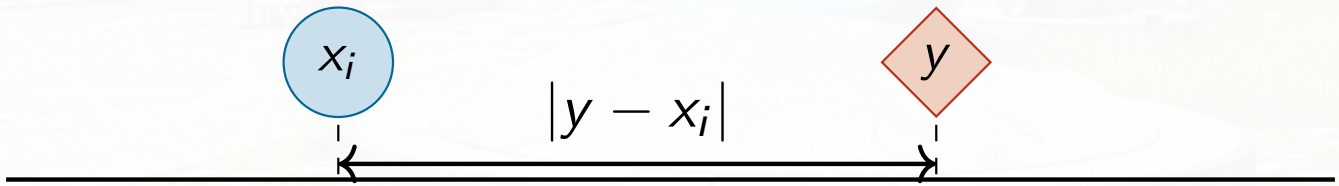

- Agent $i$ has a private location $x_i \in [0, 1]$.
- The mechanism chooses a facility location $y \in [0, 1]$.

# Utilities

## Utility of agent $i$

$$u(y, x_i) = 1 - |y - x_i|.$$

- Utility is larger when the facility is closer.
- The interval length is normalized to 1.
- The utility is always in $[0, 1]$.



# Why Utility Ratios?

A distance-ratio objective can become unbounded if the facility is exactly at an agent's location.


- Hence the paper uses utility $1 - |y - x_i|$.
- Envy ratio compares utility levels, not raw distances.

# Envy Ratio

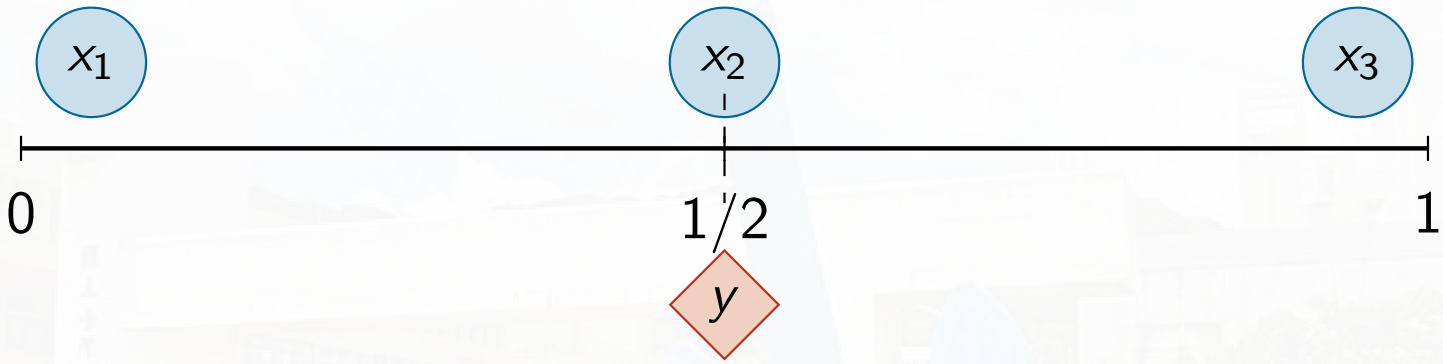
## Definition (Envy ratio)

*Given a location profile* $\mathbf{x}$ *and a facility location* $y$,

$$ER(y, \mathbf{x}) = \max_{i \neq j} \frac{u(y, x_i)}{u(y, x_j)}.$$

- The numerator is the utility of a best-off agent.
- The denominator is the utility of a worst-off agent.
- Lower is better; the ideal value is 1.

# Example: Best-off vs. Worst-off




center agent has utility 1;
endpoint agents have utility 1/2

$$\mathrm{ER}(y, \mathbf{x}) = \frac{1}{1/2} = 2.$$

# Optimal Envy Ratio

Let

$$\mathrm{lm}(\mathbf{x}) = \min_i x_i, \qquad \mathrm{rm}(\mathbf{x}) = \max_i x_i.$$

**Theorem (Ding et al. 2020)**

*The facility location that minimizes envy ratio is*

$$mid(\mathbf{x}) = \frac{lm(\mathbf{x}) + rm(\mathbf{x})}{2}.$$

# Approximation Ratio

**Definition (Approximation ratio)**

*A mechanism $f$ has approximation ratio $\rho$ if*

$$\rho = \sup_{\mathbf{x} \in [0,1]^n} \frac{ER(f(\mathbf{x}), \mathbf{x})}{ER(OPT(\mathbf{x}), \mathbf{x})}.$$

- Here $\text{OPT}(\mathbf{x}) = \text{mid}(\mathbf{x})$ for envy ratio.
- Since OPT is best possible, $\rho \geq 1$.

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- Here $\text{OPT}(\mathbf{x}) = \text{mid}(\mathbf{x})$ for envy ratio.
- Since OPT is best possible, $\rho \geq 1$.
- The challenge: require strategyproofness as well.

# Strategyproofness

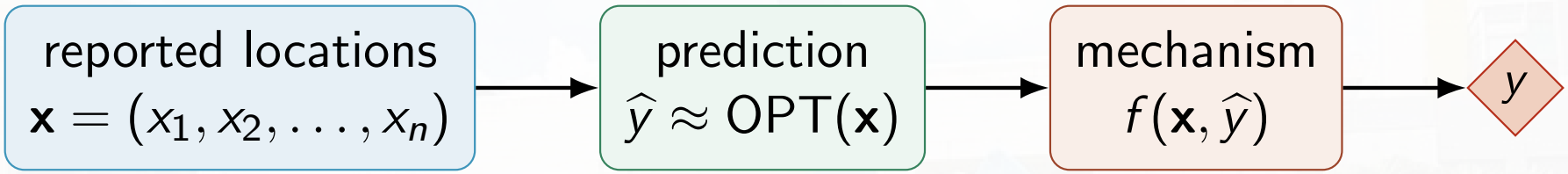
**Definition (Strategyproofness)**

*A mechanism $f$ is strategyproof if no agent can improve by misreporting:*

$$u(f(\mathbf{x}), x_i) \geq u(f(\mathbf{x}_{-i}, x_i'), x_i)$$

*for every profile $\mathbf{x}$, agent $i$, and false report $x_i'$ of agent $i$.*

# Learning-Augmented Mechanism Design




- Prediction may come from historical data or a machine-learning model.
- **Goal:** exploit accurate predictions, but retain guarantees under inaccurate predictions (robustness).

# Consistency and Robustness

$$\rho(\mathbf{x}, \hat{y}) = \frac{\mathrm{ER}(f(\mathbf{x}, \hat{y}), \mathbf{x})}{\mathrm{ER}(\mathrm{OPT}(\mathbf{x}), \mathbf{x})}$$

## $\gamma$-consistency

Approximation ratio when the prediction is correct:

$$\hat{y} = \mathrm{OPT}(\mathbf{x}).$$

That is,

$$\gamma = \sup_{\mathbf{x} \in [0,1]^n} \rho(\mathbf{x}, \mathrm{OPT}(\mathbf{x})).$$

## $\beta$-robustness

Worst-case approximation ratio under arbitrary prediction:

$$\hat{y} \in [0, 1].$$

That is,

$$\beta = \sup_{\substack{\mathbf{x} \in [0,1]^n \\ \hat{y} \in [0,1]}} \rho(\mathbf{x}, \hat{y}).$$

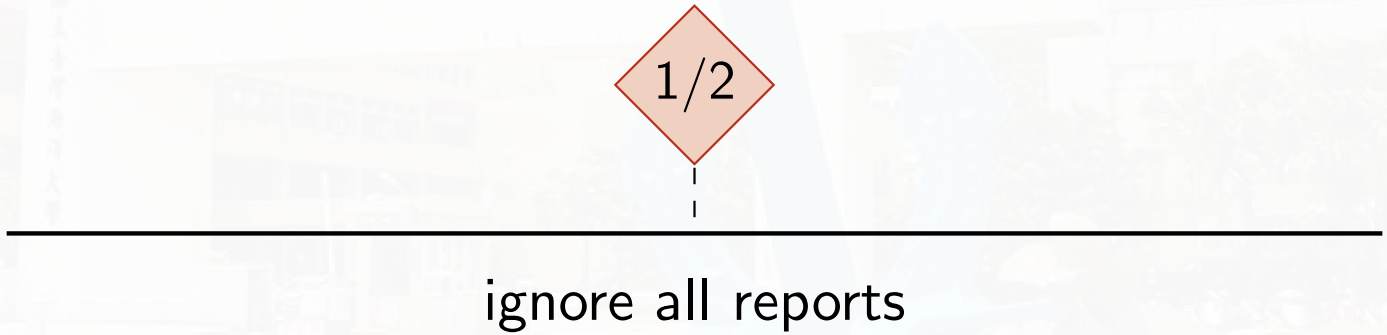
# Outline



# Baseline Without Predictions [Ding et al. 2020]

For deterministic strategyproof mechanisms without predictions:

- The **constant** mechanism $f(x) = 1/2$ is strategyproof.
- The approximation ratio 2 is best possible [Ding et al. 2020].




**Question**

Can predictions improve the consistency–robustness trade-off?

# $\alpha$-Bounding Interval Mechanism

(Extending Ding et al.'s result by considering predictions)

Choose a parameter $\alpha \in [1, 2]$.

## $\alpha$-BIM ($\alpha$-Bounding Interval Mechanism)

Define the trusted interval

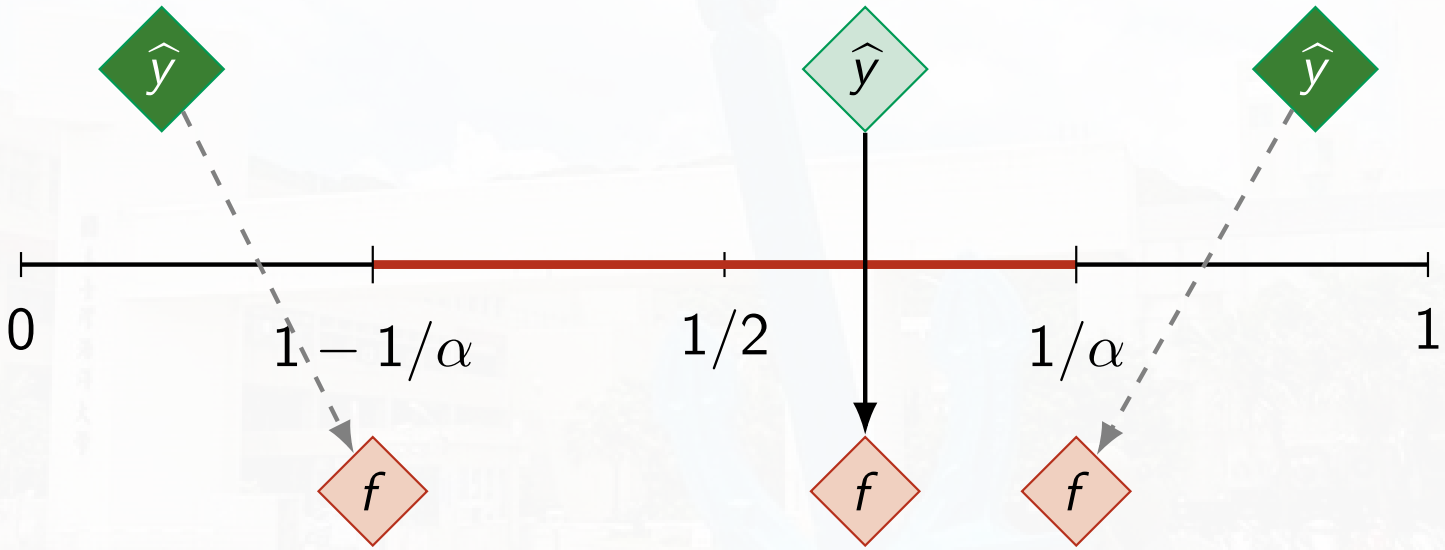
$$\left[1 - \frac{1}{\alpha}, \ \frac{1}{\alpha}\right].$$

Return $\widehat{y}$ if it lies in this interval; otherwise return the nearest endpoint.

$$f(\mathbf{x}, \widehat{y}) = \Pi_{[1-1/\alpha, 1/\alpha]}(\widehat{y}),$$

where $\Pi_I$ denotes projection onto interval $I$.

# Diagram of $\alpha$-BIM




- Inside the interval: trust the prediction.
- Outside the interval: clip it to protect robustness.

# Extreme Choices of $\alpha$

| Parameter | Trusted interval | Behavior |
|---|---|---|
| $\alpha = 1$ | $[0, 1]$ | always trust $\widehat{y}$ |
| $\alpha = 2$ | $[1/2, 1/2]$ | always output $1/2$ |

- Smaller $\alpha$: more prediction-sensitive.
- Larger $\alpha$: more conservative.

# Why Is $\alpha$-BIM Strategyproof?

**Key observation**

The output of $\alpha$-BIM depends only on the prediction $\widehat{y}$ and parameter $\alpha$, **not** on the reported locations **x**.

- If an agent changes her reported location, the output remains the same.
- Therefore no misreport can improve her utility.
- The mechanism is also anonymous: agent labels do not matter.

# Reduction to Two Agents

Let's simplify the space of location profiles which need to be considered.

**Lemma 3.1**

For any instance $\mathbf{x}$ and any distribution $P$ of facility locations, there always exists a 2-agent instance $\mathbf{x}' = (\mathsf{lm}(\mathbf{x}), \mathsf{rm}(\mathbf{x}))$ such that

$$\mathbb{E}_{y \in P}\left[\frac{\mathsf{ER}(y, \mathbf{x})}{\mathsf{ER}(\mathsf{mid}(\mathbf{x}), \mathbf{x})}\right] \leq \mathbb{E}_{y \in P}\left[\frac{\mathsf{ER}(y, \mathbf{x}')}{\mathsf{ER}(\mathsf{mid}(\mathbf{x}'), \mathbf{x}')}\right].$$

- For analyzing the approximation ratio, it suffices to consider the two extreme agents $\mathbf{x}' = (\mathsf{lm}(\mathbf{x}), \mathsf{rm}(\mathbf{x}))$.

# Proof of Lemma 3.1: Setup

- Sort the agents:
$$x_1 \leq x_2 \leq \cdots \leq x_n.$$
- Let
$$m = \text{mid}(\mathbf{x}) = \frac{x_1 + x_n}{2}.$$
- Let $u^{**}$ and $u^*$ be the maximum and minimum utilities *under the midpoint mechanism*.
$$\text{ER}(m, \mathbf{x}) = \frac{u^{**}}{u^*}.$$
- $y$: output of the mechanism.

# Proof of Lemma 3.1: Case 1

**Case 1: $y \in [x_1, x_n]$**

Let $\Delta = |y - \text{mid}(\mathbf{x})|$. When the facility moves from $\text{mid}(\mathbf{x})$ to $y$:

$$\text{maximum utility} \leq u^{**} + \Delta, \quad \text{minimum utility} \geq u^{*} - \Delta.$$

Therefore,

$$\frac{\text{ER}(y, \mathbf{x})}{\text{ER}(\text{mid}(\mathbf{x}), \mathbf{x})} \leq \frac{(u^{**} + \Delta)/(u^{*} - \Delta)}{u^{**}/u^{*}}.$$

Since the right-hand side is non-increasing in $u^{**}$,

$$\frac{(u^{**} + \Delta)/(u^{*} - \Delta)}{u^{**}/u^{*}} \leq \frac{u^{*} + \Delta}{u^{*} - \Delta}.$$

$$\boxed{\frac{\text{ER}(y, \mathbf{x})}{\text{ER}(\text{mid}(\mathbf{x}), \mathbf{x})} \leq \frac{u^{*} + \Delta}{u^{*} - \Delta}}$$

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Since the right-hand side is non-increasing in $u^{**}$, (so let's set $u^{**} = u^*$)

$$\frac{(u^{**} + \Delta)/(u^* - \Delta)}{u^{**}/u^*} \leq \frac{u^* + \Delta}{u^* - \Delta}.$$

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# Proof of Lemma 3.1: Reduction to Two Agents

- Now consider the 2-agent instance: $\mathbf{x}' = (x_1, x_n)$.
- For this instance, the midpoint is the same: $\text{mid}(\mathbf{x}') = \text{mid}(\mathbf{x})$.

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- Moreover, under the midpoint mechanism, both endpoint agents receive utility $u^*$, so $\text{ER}(\text{mid}(\mathbf{x}'), \mathbf{x}') = 1$.

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- Moreover, under the midpoint mechanism, both endpoint agents receive utility $u^*$, so $\text{ER}(\text{mid}(\mathbf{x}'), \mathbf{x}') = 1$.
- For $y \in [x_1, x_n]$, the two endpoint utilities become $u^* + \Delta$ and $u^* - \Delta$.
- Hence,

$$\frac{\text{ER}(y, \mathbf{x}')}{\text{ER}(\text{mid}(\mathbf{x}'), \mathbf{x}')} = \frac{u^* + \Delta}{u^* - \Delta}.$$

- Combining with the previous slide gives

$$\boxed{\frac{\text{ER}(y, \mathbf{x})}{\text{ER}(\text{mid}(\mathbf{x}), \mathbf{x})} \leq \frac{\text{ER}(y, \mathbf{x}')}{\text{ER}(\text{mid}(\mathbf{x}'), \mathbf{x}')}}.$$

# Proof of Lemma 3.1: Case 2 and Conclusion

## Case 2: $y \notin [x_1, x_n]$

By symmetry, assume $y > x_n$. Then $x_1$ has min. utility and $x_n$ has max. utility.

Since $d(\text{mid}(\mathbf{x}), x_1) = d(\text{mid}(\mathbf{x}), x_n) = 1 - u^*$, we have

$$u(y, x_1) = -y + \text{mid}(\mathbf{x}) + u^* \text{ and } u(y, x_n) = 2 - y + \text{mid}(\mathbf{x}) - u^*.$$

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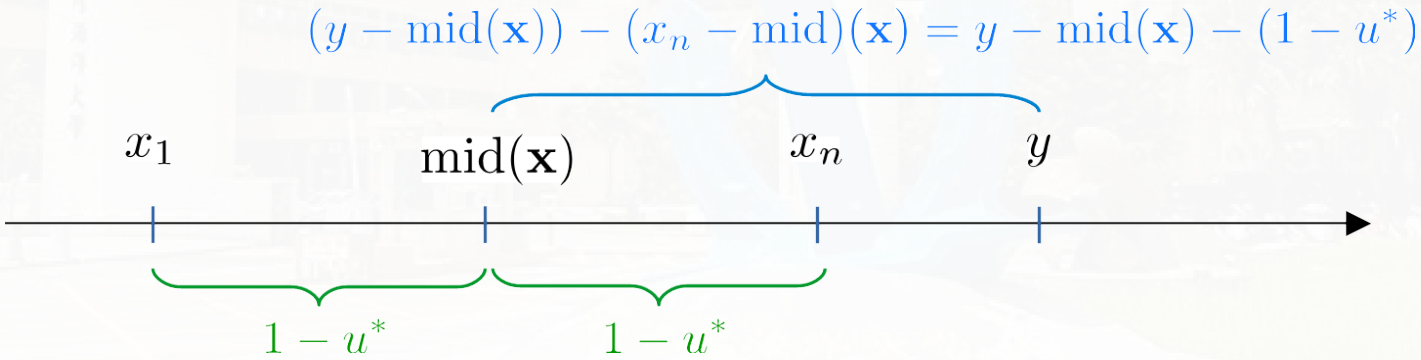

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The same expression as that for the 2-agent instance $\mathbf{x}' = (x_1, x_n)$.
Therefore, for every $y \in P$,

$$\frac{\text{ER}(y, \mathbf{x})}{\text{ER}(\text{mid}(\mathbf{x}), \mathbf{x})} \leq \frac{\text{ER}(y, \mathbf{x}')}{\text{ER}(\text{mid}(\mathbf{x}'), \mathbf{x}')}.$$

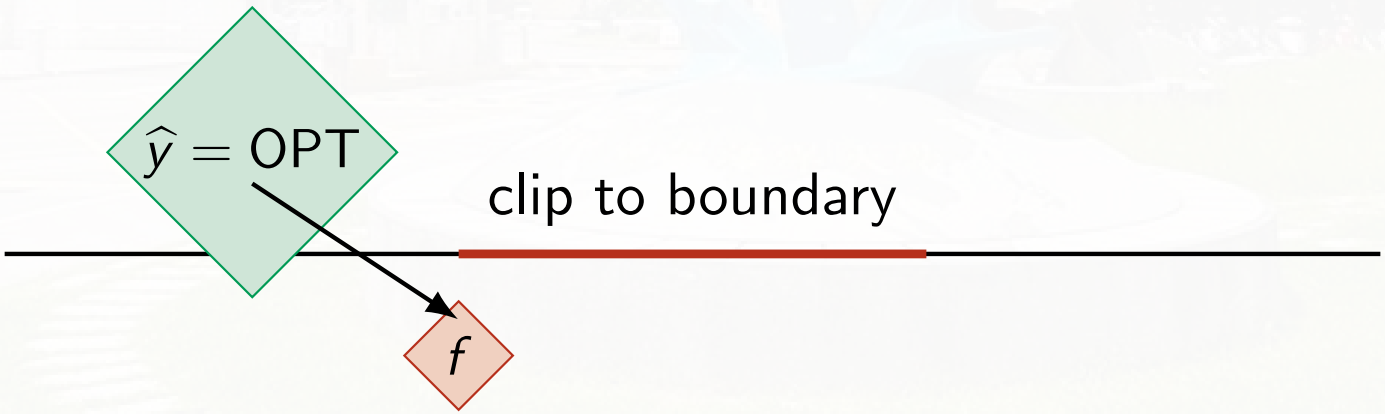
Taking expectation over $y \in P$ proves the lemma.

# Consistency of $\alpha$-BIM

**Theorem**

*When $\widehat{y} = OPT(\mathbf{x})$, $\alpha$-BIM achieves $\alpha$-consistency.*

- If the optimal midpoint lies inside the trusted interval, the mechanism is optimal.
- If it lies outside, clipping moves the facility to the nearest endpoint.
- The worst-case utility loss is bounded by a factor $\alpha$.



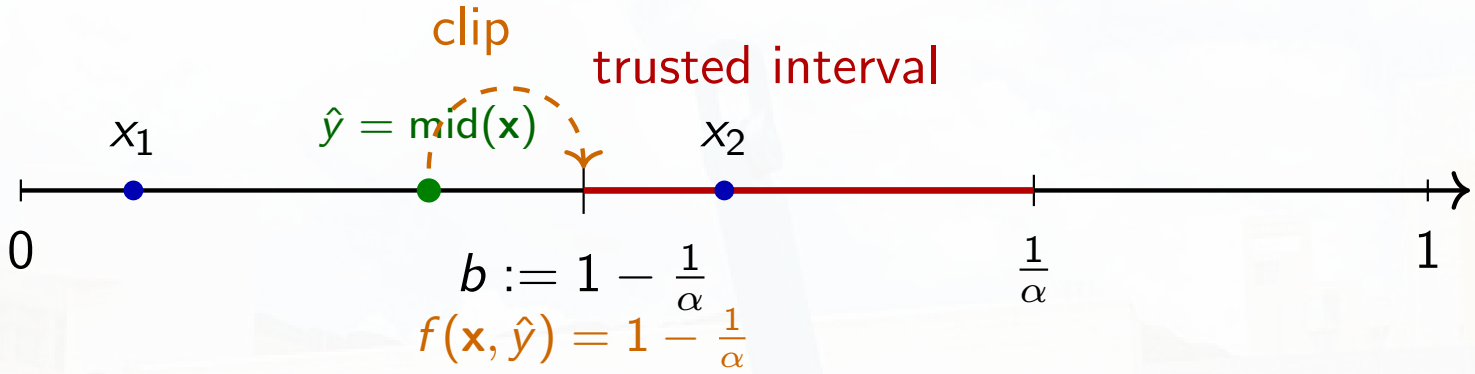
# Consistency of $\alpha$-BIM

- By Lemma 3.1, it suffices to consider a two-agent instance $\mathbf{x} = (x_1, x_2)$, $x_1 < x_2$, with $\text{mid}(\mathbf{x}) = \frac{x_1 + x_2}{2}$.
- Assume that the prediction is accurate: $\hat{y} = \text{mid}(\mathbf{x})$.
  - If $\hat{y} \in \left[1 - \frac{1}{\alpha}, \frac{1}{\alpha}\right]$, then $\alpha$-BIM returns $f(\mathbf{x}, \hat{y}) = \hat{y} = \text{mid}(\mathbf{x})$, so the mechanism is optimal on this instance.
  - Otherwise, by symmetry, it suffices to consider $\hat{y} = \text{mid}(\mathbf{x}) < 1 - \frac{1}{\alpha}$.
  - Let $b := 1 - \frac{1}{\alpha}$. In the nontrivial case,

$$x_1 < \hat{y} = \text{mid}(\mathbf{x}) < b,$$

and $\alpha$-BIM clips the prediction to $f(\mathbf{x}, \hat{y}) = b$.

# Consistency of $\alpha$-BIM: The Clipped Case




- Because $b > \text{mid}(\mathbf{x})$, we have $|b - x_2| \leq |b - x_1|$.

$\Rightarrow$ $u(b, x_2) \geq u(b, x_1)$, so $x_2$ attains the max. utility and $x_1$ the min. utility.

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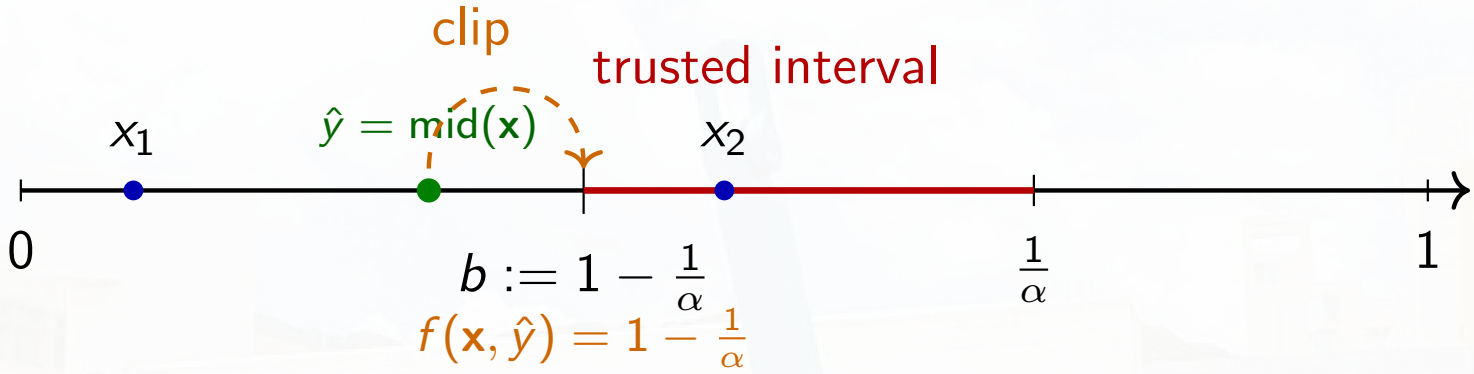


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- Moreover, $u(b, x_2) \leq 1$, $\quad u(b, x_1) = 1 - (b - x_1) \geq 1 - b = \frac{1}{\alpha}$.

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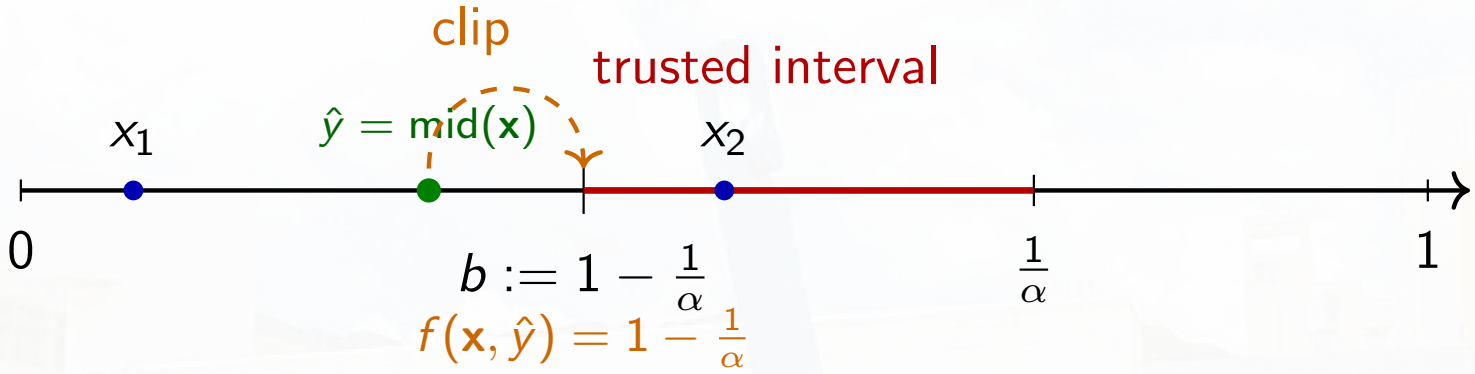


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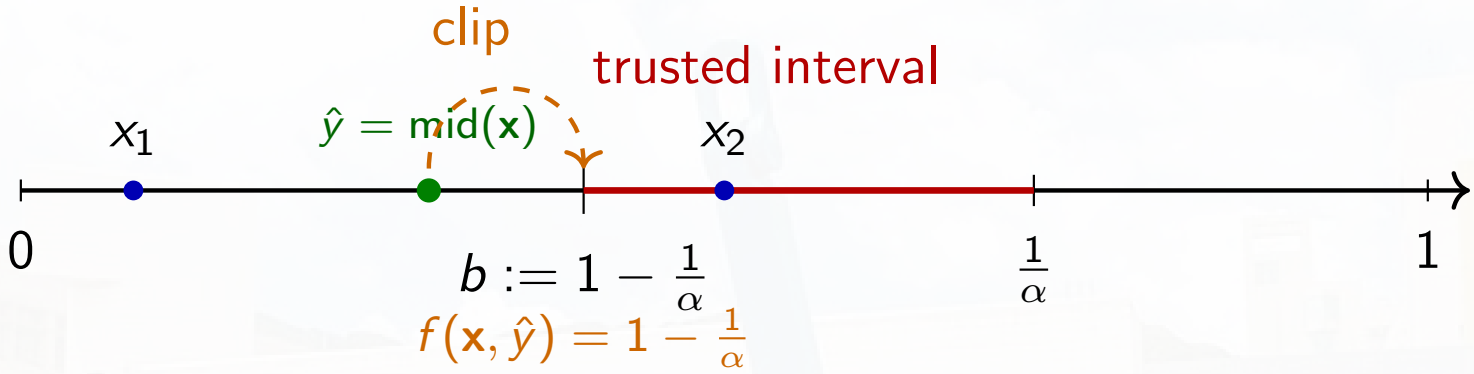


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- Moreover, $u(b, x_2) \leq 1$, $\quad u(b, x_1) = 1 - (b - x_1) \geq 1 - b = \frac{1}{\alpha}$.
- Therefore, $$\text{ER}(b, \mathbf{x}) = \frac{u(b, x_2)}{u(b, x_1)} \leq \frac{1}{1/\alpha} = \alpha.$$
- Since $\text{ER}(\text{mid}(\mathbf{x}), \mathbf{x}) = 1$, we conclude that

$$\boxed{\frac{\text{ER}(f(\mathbf{x}, \text{mid}(\mathbf{x})), \mathbf{x})}{\text{ER}(\text{mid}(\mathbf{x}), \mathbf{x})} \leq \alpha}.$$

# Robustness of $\alpha$-BIM

**Theorem**

*Under arbitrary prediction, $\alpha$-BIM achieves*

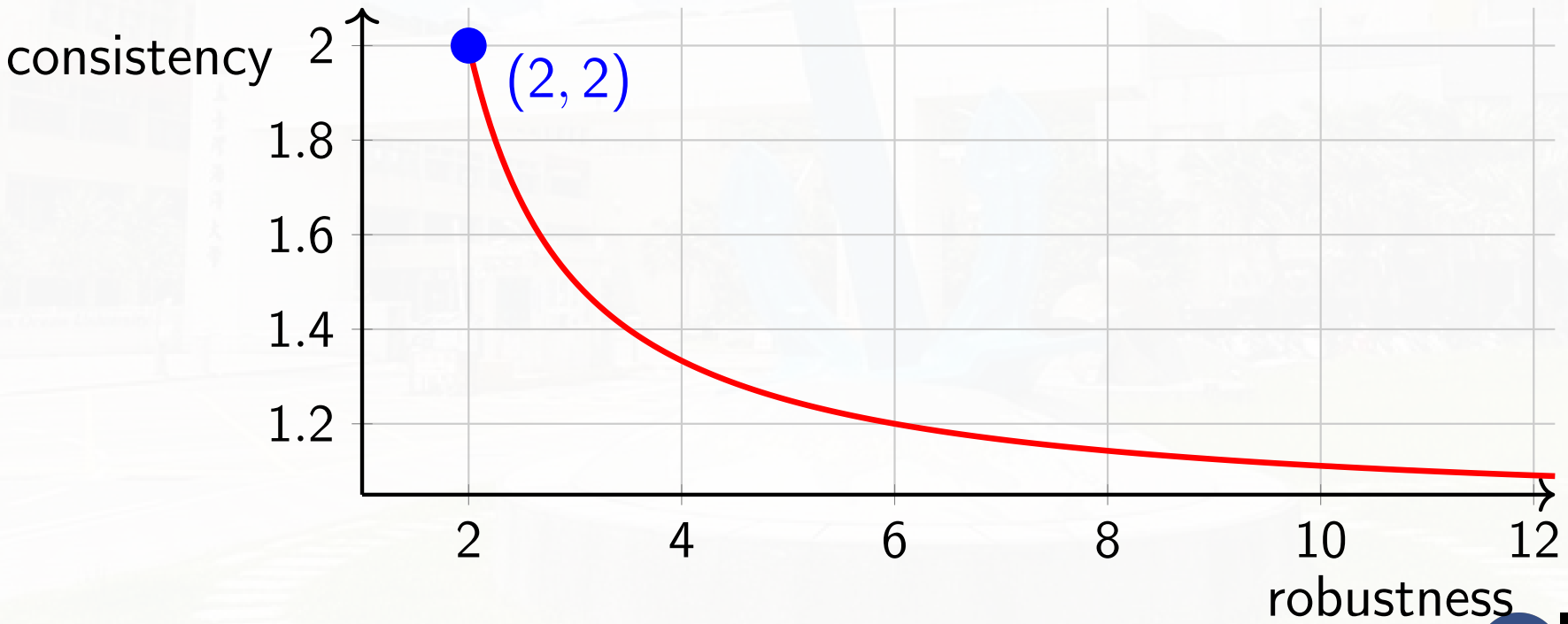
$$\frac{\alpha}{\alpha - 1}\text{-robustness.}$$

- The chosen facility is always inside $[1-1/\alpha,\ 1/\alpha]$.
- Therefore every agent has utility at least $1 - 1/\alpha$.
- The best-off utility is at most 1.

$$\mathrm{ER}(f(x, \widehat{y}), x) \leq \frac{1}{1 - 1/\alpha} = \frac{\alpha}{\alpha - 1}.$$

# Consistency–Robustness Trade-off

$$\text{consistency} = \alpha, \quad \text{robustness} = \frac{\alpha}{\alpha - 1}.$$



- Does $\alpha$-BIM achieve the "best" consistency and robustness guarantees among all Strategyproof and anonymous deterministic mechanisms?

# Optimality Among Deterministic Mechanisms

**Theorem (Optimality)**

*For every $\alpha \in (1, 2]$ and every $\varepsilon > 0$, no deterministic, strategyproof, anonymous mechanism can achieve both*

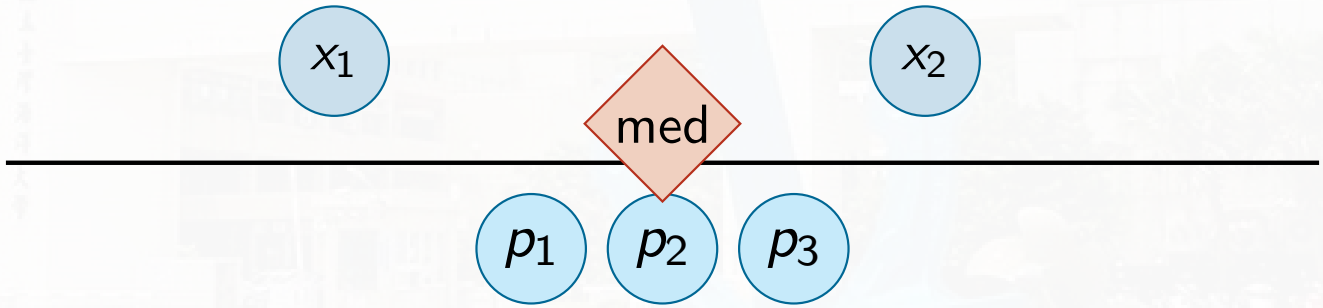
$$(\alpha - \varepsilon)\text{-consistency} \quad \text{and} \quad \left(\frac{\alpha}{\alpha - 1} - \varepsilon\right)\text{-robustness.}$$

$\alpha$-BIM lies on the deterministic Pareto frontier among all strategyproof and anonymous deterministic mechanisms.

# Proof Intuition: Phantom Mechanisms

**Proposition 2, Moulin @Public Choice 1980**

A deterministic strategyproof and anonymous mechanism on a line must be a phantom mechanism with $n+1$ constant (fixed) phantom points.




phantoms

- Robustness forces all phantoms into the BIM interval.
- Then consistency cannot beat the BIM guarantee.

## Example: A Phantom Mechanism for One Agent

Suppose the mechanism has **one** reported location $x$ and **two fixed phantom points** $a \leq b$. The output is

$$f(x) = \text{median}\{x, a, b\} = \begin{cases} a, & x < a, \\ x, & a \leq x \leq b, \\ b, & x > b. \end{cases}$$

- This is exactly a clipping rule.

# Optimality of $\alpha$-BIM: Proof Idea

Here 'phantom' locations may be a function of the prediction.

## Moulin's Characterization (Proposition 2)

Every deterministic, anonymous, and strategyproof mechanism on a line is a phantom mechanism.

For every prediction $\hat{y}$, there exist $n+1$ phantom points

$$p_1(\hat{y}), p_2(\hat{y}), \ldots, p_{n+1}(\hat{y}),$$

such that the facility location is

$$f(\mathbf{x}, \hat{y}) = \text{median}\left(x_1, x_2, \ldots, x_n, p_1(\hat{y}), p_2(\hat{y}), \ldots, p_{n+1}(\hat{y})\right).$$

- The phantom locations may depend on the prediction $\hat{y}$.
- They do *not* depend on the reported locations.

# Optimality of $\alpha$-BIM: Step 1 (in order to make $\beta \leq \alpha/(\alpha-1)$)

- Assume $1 < \alpha \leq 2$, so that $1 - \frac{1}{\alpha} \leq \frac{1}{\alpha}$.
- We show that every phantom point must satisfy

$$p_i(\hat{y}) \in \left[1 - \frac{1}{\alpha}, \frac{1}{\alpha}\right].$$

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$$p_i(\hat{y}) \in \left[1 - \frac{1}{\alpha}, \frac{1}{\alpha}\right].$$

- Suppose, for contradiction, $p_i < 1 - \frac{1}{\alpha}$. Since $p_i$ is fixed for this prediction, consider the (adversarial) location profile

$$\underbrace{p_i, \ldots, p_i}_{n-1 \text{ agents}}, \quad 1.$$

- Because $p_i$ is the median, $f(\mathbf{x}, \hat{y}) = p_i$.

# Optimality of $\alpha$-BIM: Step 2 (in order to make $\beta \leq \alpha/(\alpha-1)$)

- The facility is placed at $p_i$.

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- Therefore, robustness $> \dfrac{\alpha}{\alpha-1}$, contradicting the assumed robustness guarantee.
- Hence every phantom point must lie inside $\left[1-\dfrac{1}{\alpha}, \dfrac{1}{\alpha}\right]$.

# Optimality of $\alpha$-BIM: Conclusion (1/2)

- Fix the same prediction $\hat{y}$. Consider the location profile

$$\underbrace{\frac{1}{\alpha}, \frac{1}{\alpha}, \ldots, \frac{1}{\alpha}}_{n-1 \text{ agents}}, \quad 1.$$

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- **Claim:** $\gamma \geq \alpha$. (note that $\mathrm{ER}(\mathrm{OPT}(\mathbf{x}), \mathbf{x}) = 1$.)

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- **Claim:** $\gamma \geq \alpha$. (note that $\mathrm{ER}(\mathrm{OPT}(\mathbf{x}), \mathbf{x}) = 1$.)

  - The correct prediction is $\hat{y} = \mathrm{OPT}(\mathbf{x}) = \mathrm{mid}(\mathbf{x}) = \frac{1+\frac{1}{\alpha}}{2}$.
  - Thus
    $\mathrm{ER}(y := f(\mathbf{x}, \hat{y}), \mathbf{x}) = \frac{u(y, 1/\alpha)}{u(y, 1)} = \frac{1-1/\alpha+y}{y} = 1 + \frac{1-1/\alpha}{y} \geq \alpha.$

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- Since every phantom lies inside $\left[1-\frac{1}{\alpha}, \frac{1}{\alpha}\right]$, the median (i.e., $y$) must also lie inside this interval. $\Rightarrow$ $y := f(\mathbf{x}, \hat{y}) \in \left[1-\frac{1}{\alpha}, \frac{1}{\alpha}\right]$. $(y \leq 1/\alpha)$
- **Claim:** $\gamma \geq \alpha$. (note that $\mathrm{ER}(\mathrm{OPT}(\mathbf{x}), \mathbf{x}) = 1$.)
  - The correct prediction is $\hat{y} = \mathrm{OPT}(\mathbf{x}) = \mathrm{mid}(\mathbf{x}) = \frac{1+\frac{1}{\alpha}}{2}$.
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# Optimality of $\alpha$-BIM: Conclusion (2/2)

- Hence

  If robustness $\leq \frac{\alpha}{\alpha - 1}$, then consistency cannot be smaller than $\alpha$.

- Thus, $\alpha$-BIM simultaneously achieves the optimal consistency and robustness guarantees.

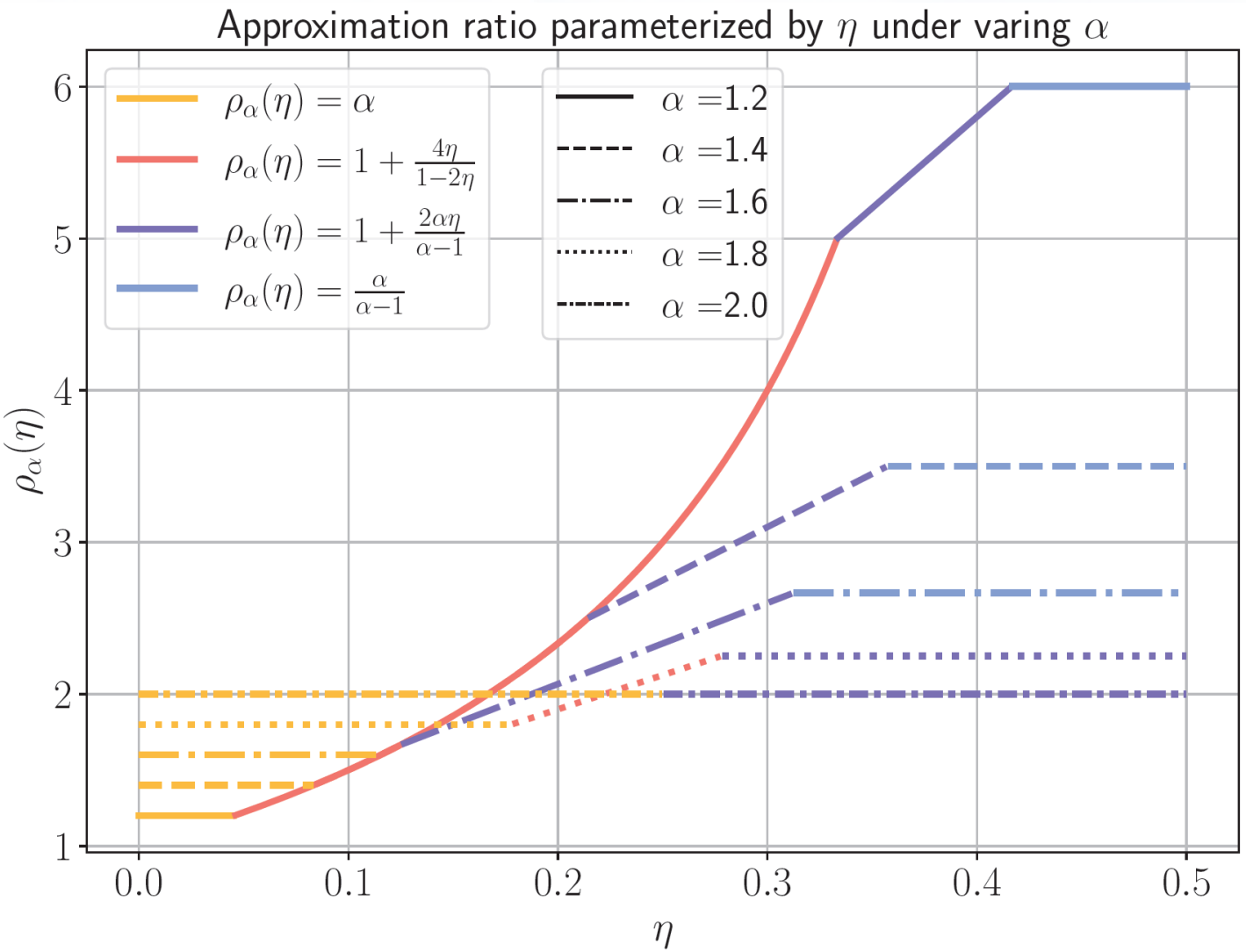
# Approximation Ratio Parameterized by Prediction Error

Let

$$y^* = \mathrm{OPT}(\mathbf{x}), \quad \hat{y} : \text{prediction}, \quad \eta = |\hat{y} - y^*|.$$

- Denote $\rho_\alpha(\eta)$: the approximation ratio as a function of prediction error $\eta$.
- The ratio increases continuously from $\alpha$ to $\alpha/(\alpha - 1)$.

# Error-Parameterized Behavior (proof skipped)



# Outline



# Why Randomization?

- Improving worst-case guarantees while preserving strategyproofness in expectation.
- Ding et al. gave a lower bound near 1.0314.
- A ratio of 2 was achievable by always outputting 1/2.
- **Open question:** Can a randomized strategyproof mechanism beat the approximation ratio of 2?

# Why Randomization?

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- A ratio of 2 was achievable by always outputting 1/2.
- **Open question:** Can a randomized strategyproof mechanism beat the approximation ratio of 2?

**This work's answer**

Yes: a new randomized mechanism achieves ratio $\approx 1.8944$.

# $(\alpha, p)$-LRM Constant Mechanism

## Mechanism: $(\alpha, p)$-LRM Constant Mechanism

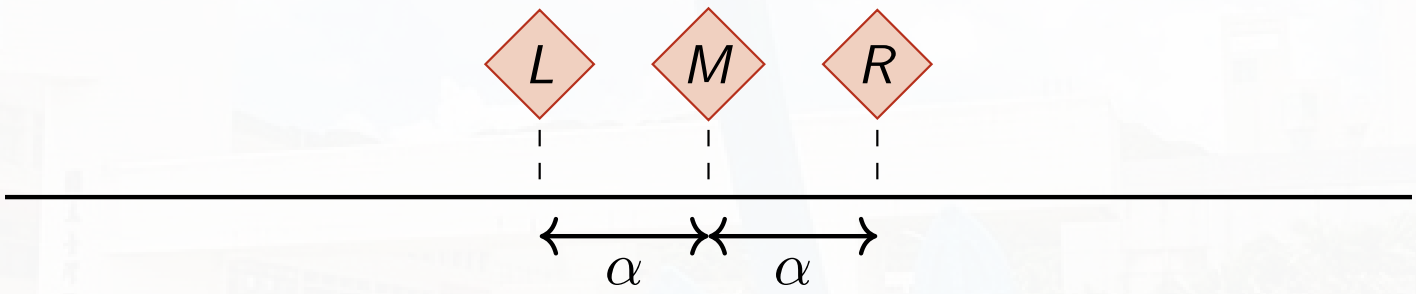
Choose parameters $\alpha \in [0, 1/2]$ and $p \in [0, 1/2]$.

$$f(\mathbf{x}) = \begin{cases} 1/2 - \alpha, & \text{with probability } p, \\ 1/2, & \text{with probability } 1 - 2p, \\ 1/2 + \alpha, & \text{with probability } p. \end{cases}$$

The mechanism is called LRM: left, right, midpoint.

▸ Go to BAM

# Diagram of LRM




probabilities: $p$, $1-2p$, $p$

- Output is **independent of reported locations**.
- Therefore it is strategyproof and anonymous.

# Choosing the Parameters

The paper proves that if $\alpha > 1/4$, the approximation ratio is already at least 2.

- Hence focus on $\alpha \leq 1/4$.
- The analysis reduces to balancing two worst-case profiles:

$$x = (0, 1/2), \qquad x' = (0, 1/2 + \alpha).$$

# Optimal LRM Parameters

## Theorem

*The optimal mechanism in the $(\alpha, p)$-LRM family uses*

$$\alpha^* = \frac{\sqrt{5}}{2} - 1, \quad p^* = \frac{2}{5}.$$

*It achieves approximation ratio*

$$1 + \frac{2}{\sqrt{5}} \approx 1.8944.$$

## Significance

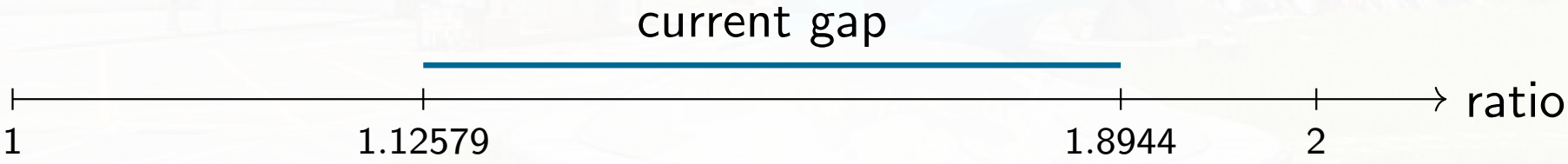
This resolves the open question of beating ratio 2 with a randomized strategyproof mechanism **without predictions**.

# Lower Bound for Randomized Mechanisms

**Theorem**

*Every randomized strategyproof mechanism without predictions has approximation ratio at least* 1.12579.

- Improves the previous lower bound of about 1.0314.
- Still far from the upper bound 1.8944.



# The main lemmas

## Lemma 4.1

When $\alpha > \frac{1}{4}$, every $(\alpha, p)$-LRM constant mechanism has an approximation ratio of at least 2.

## Lemma 4.2

Let $\mathbf{x} = \left(x_1 = 0,\ x_2 = \frac{1}{2}\right)$ and $\mathbf{x}' = \left(x_1' = 0,\ x_2' = \frac{1}{2} + \alpha\right)$. When $\alpha \leq \frac{1}{4}$, the $(\alpha^*, p^*)$-LRM constant mechanism optimizes the approximation ratio for the envy-ratio objective, where

$$(\alpha^*, p^*) = \arg\min_{(\alpha, p)} \left\{ \max \left\{ \rho(\mathbf{x}), \rho(\mathbf{x}') \right\} \right\}.$$

- Lemma 3.1 $\Rightarrow$ it suffices to consider the two extreme agents.
- **Note:** Why the two profiles $\mathbf{x}, \mathbf{x}'$? Wasn't clearly proved in the paper.

# The main theorem on Mechanisms without Predictions

## Theorem 4.3

The

$$\left(\frac{\sqrt{5}}{2}-1,\ \frac{2}{5}\right)\text{-LRM}$$

constant mechanism is anonymous and strategyproof, and achieves approximation ratio

$$1+\frac{2}{\sqrt{5}}\approx 1.8944.$$

Moreover, this ratio is optimal among all $(\alpha, p)$-LRM constant mechanisms.

Let $\alpha^* = \frac{\sqrt{5}}{2} - 1$, $p^* = \frac{2}{5}$. The mechanism selects the facility location according to

| facility location | probability |
|---|---|
| $\frac{1}{2} - \alpha^* = \frac{3-\sqrt{5}}{2}$ | $\frac{2}{5}$ |
| $\frac{1}{2}$ | $1 - 2p^* = \frac{1}{5}$ |
| $\frac{1}{2} + \alpha^* = \frac{\sqrt{5}-1}{2}$ | $\frac{2}{5}$ |

## Optimality

For the two critical profiles $\mathbf{x} = \left(0, \frac{1}{2}\right)$, $\mathbf{x}' = \left(0, \frac{1}{2} + \alpha^*\right)$, the approximation ratios are equal:

$$\rho(\mathbf{x}) = \rho(\mathbf{x}') = 1 + \frac{2}{\sqrt{5}}.$$

No other choice of $(\alpha, p)$ yields a smaller worst-case ratio.

# Proof of Lemma 4.1 (1/2)

Let the $(\alpha, p)$-LRM mechanism output

$$L = \frac{1}{2} - \alpha, \qquad C = \frac{1}{2}, \qquad R = \frac{1}{2} + \alpha$$

with probabilities $p$, $1 - 2p$, $p$, respectively.

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with probabilities $p$, $1 - 2p$, $p$, respectively. Consider the two-agent instance

$$\mathbf{x} = (x_1, x_2) = \left(0, \frac{1}{2}\right).$$

The optimal midpoint $(m)$ gives envy ratio 1 ($\because u(m, x_1) = u(m, x_2)$)).

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$$\mathbf{x} = (x_1, x_2) = \left(0, \frac{1}{2}\right).$$

The optimal midpoint $(m)$ gives envy ratio 1 ($\because u(m, x_1) = u(m, x_2)$). For $\alpha > \frac{1}{4}$, the realized envy ratios are

$$\mathrm{ER}(L, \mathbf{x}) = \frac{\frac{1}{2} + \alpha}{1 - \alpha}, \quad \mathrm{ER}(C, \mathbf{x}) = 2, \quad \mathrm{ER}(R, \mathbf{x}) = \frac{1 - \alpha}{\frac{1}{2} - \alpha}.$$

# Proof of Lemma 4.1 (2/2)

Hence

$$\rho(\mathbf{x}) = p\frac{\frac{1}{2}+\alpha}{1-\alpha} + (1-2p)2 + p\frac{1-\alpha}{\frac{1}{2}-\alpha}.$$

Both outer terms are increasing in $\alpha$ on $(\frac{1}{4}, \frac{1}{2})$. At $\alpha = \frac{1}{4}$,

$$\frac{\frac{1}{2}+\alpha}{1-\alpha} = 1, \qquad \frac{1-\alpha}{\frac{1}{2}-\alpha} = 3.$$

Therefore

$$\rho(\mathbf{x}) \geq p \cdot 1 + (1-2p)2 + p \cdot 3 = 2.$$

Thus every $(\alpha, p)$-LRM constant mechanism with $\alpha > \frac{1}{4}$ has approximation ratio at least 2.

# Proof of Lemma 4.2: Setup

Let

$$L = \frac{1}{2} - \alpha, \qquad C = \frac{1}{2}, \qquad R = \frac{1}{2} + \alpha.$$

By Lemma 3.1, it suffices to consider two-agent instances

$$\mathbf{x} = (x_1, x_2), \qquad 0 \le x_1 \le x_2 \le 1.$$

By symmetry, assume $\mathrm{mid}(\mathbf{x}) \le \frac{1}{2}$. Write $m = \mathrm{mid}(\mathbf{x})$, $\delta = \frac{1}{2} - m$.
Let $u$ be the utility of both agents when the facility is placed at $m$. Since $x_1 \ge 0$,

$$u \ge \frac{1}{2} + \delta.$$

**Goal**

Show that, for the optimal choice of $(\alpha, p)$, it is enough to compare the two profiles

$$\mathbf{x} = \left(0, \frac{1}{2}\right), \quad \mathbf{x}' = \left(0, \frac{1}{2} + \alpha\right).$$

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Let $u$ be the utility of both agents when the facility is placed at $m$. Since $x_1 \ge 0$,

$$u \ge \frac{1}{2} + \delta. \ (\text{e.g.}, 1 - (m - x_1) = (1/2) + \delta + x_1 \ge 1/2 + \delta)$$

**Goal**

Show that, for the optimal choice of $(\alpha, p)$, it is enough to compare the two profiles

$$\mathbf{x} = \left(0, \frac{1}{2}\right), \quad \mathbf{x}' = \left(0, \frac{1}{2} + \alpha\right).$$

# Why Consider Only the Two Profiles in Lemma 4.2?

By Lemma 3.1, it suffices to analyze two-agent instances. By symmetry and Claim B.1, we may further reduce to

$$\mathbf{x} = (0, t), \qquad 0 \le t \le 1.$$

Thus the problem becomes one-dimensional:

$$\rho(t) := \rho((0, t)).$$

**Goal**

Show that the worst case over all $t$ is captured by only

$$t = C = \frac{1}{2} \qquad \text{or} \qquad t = R = \frac{1}{2} + \alpha.$$

# Claim B.1: Reduction to $x_1 = 0$

## Claim B.1

Suppose

$$\text{mid}(\mathbf{x}) \in \left[0, \frac{1}{2} - \alpha\right).$$

For any two-agent instance $\mathbf{x} = (x_1, x_2)$, let $\mathbf{x}' = (0, x_2)$. Then the approximation ratio of the $(\alpha, p)$-LRM mechanism satisfies $\rho(\mathbf{x}) \leq \rho(\mathbf{x}')$.

- Thus, in this region, it suffices to analyze only instances of the form $(0, x_2)$.
- This reduces the worst-case search from two variables $(x_1, x_2)$ to a single variable $x_2$.

# Proof of Claim B.1

Since $\text{mid}(\mathbf{x}) = \frac{x_1 + x_2}{2} \leq \frac{1}{2} - \alpha$, and every possible facility location satisfies

$$y \in \left\{ \frac{1}{2} - \alpha,\ \frac{1}{2},\ \frac{1}{2} + \alpha \right\},$$

we have

$$y \geq \frac{1}{2} - \alpha \geq \text{mid}(\mathbf{x}).$$

Therefore,

$$y - x_1 \geq |y - x_2|.$$

Hence agent 1 is always the worse-off agent: $u_1(y) \leq u_2(y)$, for every $y$. The expected approximation ratio is therefore

$$\rho(\mathbf{x}) = p\frac{u_2(L)}{u_1(L)} + (1-2p)\frac{u_2(C)}{u_1(C)} + p\frac{u_2(R)}{u_1(R)}.$$

# Proof of Claim B.1 (continued)

- Move the left agent from $x_1$ to 0, obtaining $\mathbf{x}' = (0, x_2)$.
- Observe that $u_2(y)$ does not change, while $u_1(y) = 1 - (y - x_1)$ becomes

$$u_1'(y) = 1 - y \leq 1 - (y - x_1) = u_1(y).$$

- Hence

$$\frac{u_2(y)}{u_1(y)} \leq \frac{u_2(y)}{u_1'(y)} \qquad \forall\, y \in \{L, C, R\}.$$

- Taking the weighted average, we have

$$\rho(\mathbf{x}) \leq \rho(\mathbf{x}').$$

- Therefore every worst-case instance in this region may be assumed to satisfy $x_1 = 0$.

# Piecewise Monotonicity

- For fixed $(\alpha, p)$, the mechanism randomizes over $L,\ C,\ R$.
- The expression for $\rho(t)$ changes only when $t$ crosses one of $L,\ C,\ R,\ 1-2\alpha$.
- When $0 \le \alpha < \frac{1}{6}$, their order is $L < C < R < 1-2\alpha$.
- Appendix B.2 shows (proof omitted here), by differentiating $\rho(t)$, that the only endpoint candidates are

$$t = L, \quad C, \quad R, \quad 1-2\alpha.$$

So the continuous worst-case search reduces to finitely many candidates.

# Eliminating the Extra Candidates

- The appendix then compares the candidate values:

$$\rho(L) \le \rho(C), \qquad \rho(1-2\alpha) \le \rho(R).$$

- Therefore the candidates $L$ and $1-2\alpha$ are dominated.
- Hence

$$\sup_{t\in[0,1]} \rho(t) = \max\{\rho(C), \rho(R)\}.$$

Since

$$C = \frac{1}{2}, \qquad R = \frac{1}{2} + \alpha,$$

the two surviving profiles are

$$\mathbf{x} = \left(0, \frac{1}{2}\right), \qquad \mathbf{x}' = \left(0, \frac{1}{2} + \alpha\right).$$

# Meaning of Lemma 4.2

Lemma 4.2 says that optimizing over all instances is equivalent to optimizing over the two surviving worst cases:

$$\min_{\alpha,p} \sup_{\mathbf{z}} \rho(\mathbf{z}) = \min_{\alpha,p} \max \left\{ \rho\left(0, \frac{1}{2}\right), \rho\left(0, \frac{1}{2} + \alpha\right) \right\}.$$

- $\left(0, \frac{1}{2}\right)$ is one extremal constraint.
- $\left(0, \frac{1}{2} + \alpha\right)$ is the other extremal constraint.
- Every other profile has approximation ratio no larger than one of these two.

So the optimal $(\alpha, p)$ is found by balancing these two cases.

# Proof of Lemma 4.2: Case $m \in [L, C]$

- Assume

$$m = \frac{1}{2} - \delta \in \left[\frac{1}{2} - \alpha,\ \frac{1}{2}\right], \quad 0 \le \delta \le \alpha.$$

The distances from $m$ to the three possible facility locations are $\alpha - \delta,\ \delta,\ \alpha + \delta$.
Thus

$$\rho(\mathbf{x}) \le p\frac{u + \alpha - \delta}{u - (\alpha - \delta)} + (1 - 2p)\frac{u + \delta}{u - \delta} + p\frac{u + (\alpha + \delta)}{u - (\alpha + \delta)}.$$

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- Since the right-hand side is non-increasing in $u$, and $u \ge \frac{1}{2} + \delta$, we get

$$\begin{aligned}\rho(\mathbf{x}) &\le p\frac{1 + 2\alpha}{1 + 4\delta - 2\alpha} + (1 - 2p)(1 + 4\delta) + p\frac{1 + 4\delta + 2\alpha}{1 - 2\alpha} \\ &\le 2p\frac{1 + 2\alpha}{1 - 2\alpha} + (1 - 2p)(1 + 4\delta).\end{aligned}$$

# Proof of Lemma 4.2: Case $m \in [L, C]$

- Assume

$$m = \frac{1}{2} - \delta \in \left[\frac{1}{2} - \alpha,\ \frac{1}{2}\right], \quad 0 \le \delta \le \alpha.$$

The distances from $m$ to the three possible facility locations are $\alpha - \delta,\ \delta,\ \alpha + \delta$. Thus

$$\rho(\mathbf{x}) \le p\frac{u + \alpha - \delta}{u - (\alpha - \delta)} + (1 - 2p)\frac{u + \delta}{u - \delta} + p\frac{u + (\alpha + \delta)}{u - (\alpha + \delta)}.$$

- Since the right-hand side is non-increasing in $u$, and $u \ge \frac{1}{2} + \delta$, we get

$$\begin{aligned}\rho(\mathbf{x}) &\le p\frac{1 + 2\alpha}{1 + 4\delta - 2\alpha} + (1 - 2p)(1 + 4\delta) + p\frac{1 + 4\delta + 2\alpha}{1 - 2\alpha} \\ &\le 2p\frac{1 + 2\alpha}{1 - 2\alpha} + (1 - 2p)(1 + 4\delta).\end{aligned}$$

This is increasing in $\delta$. Since $\frac{1}{2} - \alpha = m = \frac{x_1 + x_2}{2} \Rightarrow x_1 + x_2 = 1 - 2\alpha$ and $x_1 = 0$, so the worst case in this region occurs at

$$\delta = \alpha, \quad \mathbf{x} = (0,\ 1 - 2\alpha).$$

# Proof of Lemma 4.2: Case $m < L$

- Now consider the case $m < L = \frac{1}{2} - \alpha$.
- For every possible facility location $y \in \{L, C, R\}$, we have $y \geq m$. Hence $y - x_1 \geq |y - x_2|$, for any $y \in \{\frac{1}{2} - \alpha, \frac{1}{2}, \frac{1}{2} + \alpha\}$.
- Therefore agent $x_1$ is weakly worse off than agent $x_2$ for every realized facility location $y$.

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- Therefore agent $x_1$ is weakly worse off than agent $x_2$ for every realized facility location $y$.
- Recall that we only need to consider $\mathbf{x} = (0, x_2)$ and let $t := x_2$. Then every remaining instance can be written as $\mathbf{x} = (0, t)$, $0 \leq t \leq 1$.
- For convenience, define $\rho(t) := \rho((0, t))$.

# Proof of Lemma 4.2: The Range $\alpha < 1/6$

- Assume $0 \le \alpha < \frac{1}{6}$. Then $L < C < R < 1 - 2\alpha$.
- For $\mathbf{x} = (0, t)$, direct differentiation gives the following monotonicity table.

| range of $t$ | sign / behavior of $\rho'(t)$ | candidate worst case |
|---|---|---|
| $[0, L]$ | $\frac{2p}{1+2\alpha} + 2(1-2p) + \frac{2p}{1-2\alpha} \ge 0$ | $t = L$ |
| $(L, C]$ | $\frac{8p\alpha}{1-4\alpha^2} + 2(1-2p) \ge 0$ | $t = C$ |
| $(C, R]$ | $D = \frac{8p\alpha}{1-4\alpha^2} + 4p - 2$ constant | $t = C$ or $t = R$ |
| $(R, 1-2\alpha]$ | $-\frac{4p}{1-4\alpha^2} - 2(1-2p) \le 0$ | $t = R$ |

Thus the only candidates are

$$t = L, \quad C, \quad R, \quad 1 - 2\alpha.$$

Direct substitution gives

$$\rho(L) \le \rho(C), \qquad \rho(1-2\alpha) \le \rho(R).$$

Hence, when $\alpha < 1/6$, the worst cases reduce to

$$(0, C) = \left(0, \frac{1}{2}\right) \quad \text{and} \quad (0, R) = \left(0, \frac{1}{2} + \alpha\right).$$

# Why is $\frac{1}{6}$ the breakpoint?

- If we have assumed $L < C < R < 1 - 2\alpha$, then $R = \frac{1}{2} + \alpha < 1 - 2\alpha \iff \alpha < \frac{1}{6}$.
- For $\alpha > \frac{1}{6}$, $R > 1 - 2\alpha$.

# Proof of Lemma 4.2: Excluding $\alpha \geq 1/6$

- For the two candidate profiles, define $F_\alpha(p) := \rho\big(0, \frac{1}{2}\big)$, $G_\alpha(p) := \rho\big(0, \frac{1}{2} + \alpha\big)$.
- When $\alpha \in [1/6, 1/4]$, direct calculation gives $F_\alpha(p) = 2(1-2p) + p\dfrac{4-4\alpha}{1-4\alpha^2}$, and $G_\alpha(p) = p\dfrac{1+2\alpha}{2-4\alpha} + (1-2p)(2-2\alpha) + p\dfrac{2}{1-2\alpha}$. (Note: $F'_\alpha(p) \leq 0$, $G_\alpha(p) > 0$.)
- For fixed $\alpha$, the best $p$ equalizes the two affine functions: $F_\alpha(p) = G_\alpha(p)$.

  Solving this gives $p = \dfrac{16\alpha^3 - 4\alpha}{32\alpha^3 - 4\alpha^2 - 28\alpha + 3}$.
- Then the substitution yields

$$F_\alpha(p) = G_\alpha(p) = \frac{8\alpha^2 - 56\alpha + 6}{32\alpha^3 - 4\alpha^2 - 28\alpha + 3}.$$

- This expression is increasing on $\alpha \in \left[\frac{1}{6}, \frac{1}{4}\right]$. Therefore its minimum is attained at $\alpha = \frac{1}{6}$, $p = \frac{4}{11}$, with value $\frac{21}{11} \approx 1.909$.

# Proof of Lemma 4.2: Conclusion (1/2)

- From the previous slide, every mechanism with $\alpha \in \left[\frac{1}{6}, \frac{1}{4}\right]$ has approximation ratio at least
$$\frac{21}{11} \approx 1.909$$
on one of the two profiles $\left(0, \frac{1}{2}\right), \ \left(0, \frac{1}{2} + \alpha\right)$.
- But for
$$\alpha^* = \frac{\sqrt{5}}{2} - 1 < \frac{1}{6}, \quad p^* = \frac{2}{5},$$
the two-profile maximum is
$$1 + \frac{2}{\sqrt{5}} \approx 1.8944.$$
- Hence no globally optimal $(\alpha, p)$ lies in $\left[\frac{1}{6}, \frac{1}{4}\right]$.

# Proof of Lemma 4.2: Conclusion (2/2)

- Therefore the global optimum lies in the range $\alpha < \frac{1}{6}$, where the previous reduction shows that the only worst-case profiles are $\left(0, \frac{1}{2}\right)$ and $\left(0, \frac{1}{2} + \alpha\right)$.
- Thus the optimal parameters can be obtained by solving

$$(\alpha^*, p^*) = \arg\min_{\alpha, p} \left\{ \max \left\{ \rho\left(0, \frac{1}{2}\right), \rho\left(0, \frac{1}{2} + \alpha\right) \right\} \right\}.$$

# Proof of Theorem 4.3: Setup

**Theorem 4.3**

The $\left(\frac{\sqrt{5}}{2}-1, \frac{2}{5}\right)$-LRM constant mechanism is anonymous, strategyproof, and achieves approximation ratio

$$1+\frac{2}{\sqrt{5}} \approx 1.8944.$$

- The mechanism returns

$$\frac{1}{2}-\alpha, \quad \frac{1}{2}, \quad \frac{1}{2}+\alpha$$

with probabilities $p$, $1-2p$, $p$, resp.

- It is anonymous and strategyproof because the output distribution is independent of the agents' reports.
- By Lemma 4.2, it suffices to solve $\min_{\alpha,p} \max\{\rho(\mathbf{x}), \rho(\mathbf{x}')\}$, where $\mathbf{x}=\left(0, \frac{1}{2}\right)$, $\mathbf{x}'=\left(0, \frac{1}{2}+\alpha\right)$.

# Proof of Theorem 4.3: Ratio on $\mathbf{x}$

- For $\mathbf{x} = \left(0, \frac{1}{2}\right)$, the approximation ratio is

$$\rho(\mathbf{x}) = p \cdot \frac{1-\alpha}{1-(\frac{1}{2}-\alpha)} + (1-2p) \cdot \frac{1}{1-\frac{1}{2}} + p \cdot \frac{1-\alpha}{1-(\frac{1}{2}+\alpha)}.$$

- Hence

$$\rho(\mathbf{x}) = p \cdot \frac{1-\alpha}{\frac{1}{2}+\alpha} + 2(1-2p) + p \cdot \frac{1-\alpha}{\frac{1}{2}-\alpha}.$$

- Equivalently,

$$\rho(\mathbf{x}) = 2(1-2p) + p \cdot \frac{4-4\alpha}{1-4\alpha^2}.$$

# Proof of Theorem 4.3: Ratio on $\mathbf{x}'$

For

$$\mathbf{x}' = \left(0, \frac{1}{2} + \alpha\right),$$

there are two parameter regimes.

**Case 1: $\alpha \in [0, \frac{1}{6})$**

$$\rho(\mathbf{x}') = p \cdot \frac{2 - 4\alpha}{1 + 2\alpha} + (1 - 2p)(2 - 2\alpha) + p \cdot \frac{2}{1 - 2\alpha}.$$

**Case 2: $\alpha \in [\frac{1}{6}, \frac{1}{4}]$**

$$\rho(\mathbf{x}') = p \cdot \frac{1 + 2\alpha}{2 - 4\alpha} + (1 - 2p)(2 - 2\alpha) + p \cdot \frac{2}{1 - 2\alpha}.$$

**Note:** $u_1(L) = 1 - ((\frac{1}{2} - \alpha) - 0) = \frac{1}{2} + \alpha$, $u_2(L) = 1 - ((\frac{1}{2} + \alpha) - (\frac{1}{2} - \alpha)) = 1 - 2\alpha$.

Thus we solve the minimax problem separately in the two regimes.

# Proof of Theorem 4.3: Case $\alpha \in [0, \frac{1}{6})$

To minimize

$$\max\{\rho(\mathbf{x}), \rho(\mathbf{x}')\},$$

the optimum equalizes the two expressions:

$$\rho(\mathbf{x}) = \rho(\mathbf{x}').$$

Using the formulas from the previous slides gives

$$2(1-2p) + p \cdot \frac{4-4\alpha}{1-4\alpha^2} = p \cdot \frac{2-4\alpha}{1+2\alpha} + (1-2p)(2-2\alpha) + p \cdot \frac{2}{1-2\alpha}.$$

Solving for $p$, we get

$$p = \frac{4\alpha^2 - 1}{2(4\alpha^2 - 2\alpha - 1)}.$$

Substituting this into $\rho(\mathbf{x})$ gives

$$\rho(\mathbf{x}) = \rho(\mathbf{x}') = \frac{2\alpha + 2}{-4\alpha^2 + 2\alpha + 1}.$$

# Proof of Theorem 4.3: Optimizing Case 1

We now minimize

$$R(\alpha) = \frac{2\alpha + 2}{-4\alpha^2 + 2\alpha + 1}$$

over

$$\alpha \in \left[0, \frac{1}{6}\right).$$

Its derivative is

$$R'(\alpha) = \frac{8\alpha^2 + 16\alpha - 2}{(-4\alpha^2 + 2\alpha + 1)^2}.$$

Thus the unique nonnegative critical point is

$$\alpha^* = \frac{\sqrt{5}}{2} - 1.$$

Substituting into the formula for $p$,

$$p^* = \frac{4(\alpha^*)^2 - 1}{2(4(\alpha^*)^2 - 2\alpha^* - 1)} = \frac{2}{5}.$$

Therefore

$$R(\alpha^*) = 1 + \frac{2}{\sqrt{5}} \approx 1.8944.$$

# Proof of Theorem 4.3: Case $\alpha \in [\frac{1}{6}, \frac{1}{4}]$

Now use the second formula for $\rho(\mathbf{x}')$:

$$\rho(\mathbf{x}') = p \cdot \frac{1+2\alpha}{2-4\alpha} + (1-2p)(2-2\alpha) + p \cdot \frac{2}{1-2\alpha}.$$

Again equalize

$$\rho(\mathbf{x}) = \rho(\mathbf{x}').$$

Solving gives

$$p = \frac{16\alpha^3 - 4\alpha}{32\alpha^3 - 4\alpha^2 - 28\alpha + 3}.$$

Substituting back gives

$$\rho(\mathbf{x}) = \rho(\mathbf{x}') = \frac{8\alpha^2 - 56\alpha + 6}{32\alpha^3 - 4\alpha^2 - 28\alpha + 3}.$$

This expression is increasing on

$$\left[\frac{1}{6}, \frac{1}{4}\right].$$

Thus the best value in this regime occurs at

$$\alpha = \frac{1}{6}, \qquad p = \frac{4}{11}.$$

# Proof of Theorem 4.3: Comparing the Two Regimes

- For $\alpha \in \left[0, \frac{1}{6}\right)$, the optimal parameters are

$$\alpha^* = \frac{\sqrt{5}}{2} - 1, \quad p^* = \frac{2}{5},$$

with approximation ratio $1 + \frac{2}{\sqrt{5}} \approx 1.8944$.

- For $\alpha \in \left[\frac{1}{6}, \frac{1}{4}\right]$, the best parameters are

$$\alpha = \frac{1}{6}, \quad p = \frac{4}{11},$$

with approximation ratio $\frac{21}{11} \approx 1.909$.

- Since $1 + \frac{2}{\sqrt{5}} < \frac{21}{11}$, the global optimum among all $(\alpha, p)$-LRM constant mechanisms is

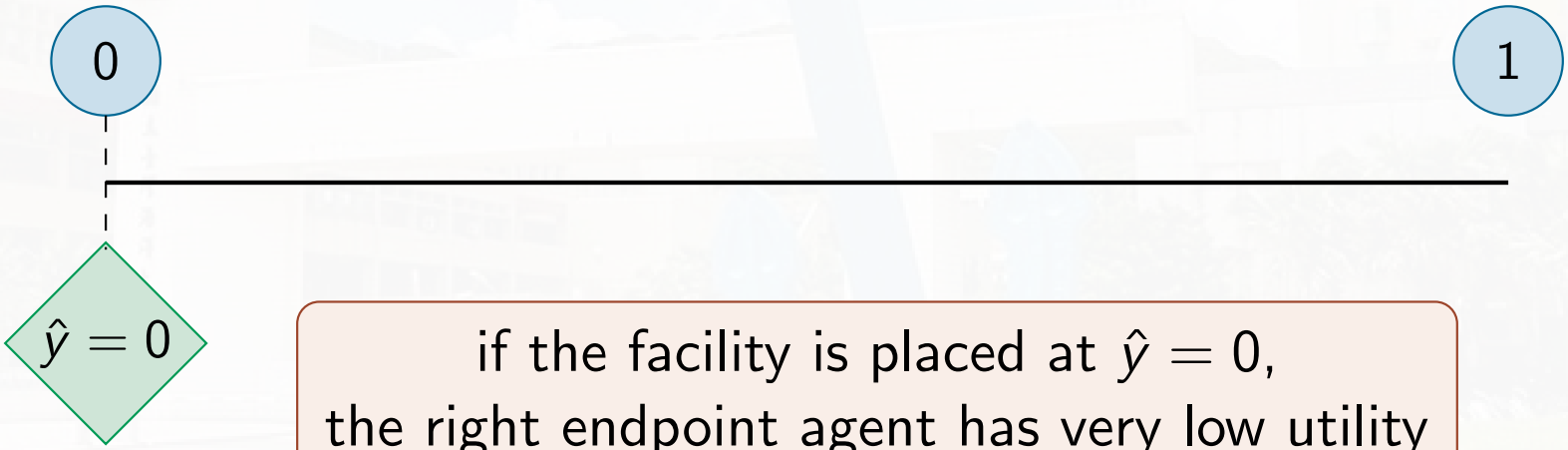
$$(\alpha^*, p^*) = \left(\frac{\sqrt{5}}{2} - 1, \frac{2}{5}\right).$$

# Outline



# The Main Difficulty

- ⋆ A prediction-sensitive randomized mechanism might put positive probability on $\hat{y}$.




- More trust in $\hat{y}$ helps consistency.
- Too much trust can destroy robustness.

# Bias-Aware Mechanism: Idea

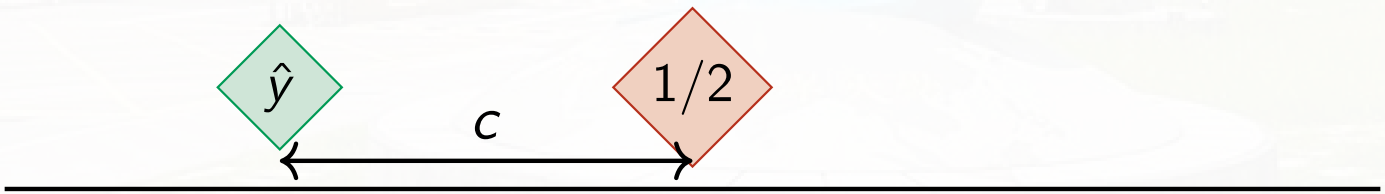
- Let

$$c = \left| \hat{y} - \frac{1}{2} \right|.$$

This is the bias of the prediction away from the center.

**Design principle**

Trust the prediction less when it is closer to an endpoint.



# Bias-Aware Mechanism (BAM)

## Mechanism: Bias-Aware Mechanism (BAM)

Given prediction $\widehat{y}$, compute

$$c = \left|\hat{y} - \frac{1}{2}\right|, \quad p = \frac{1}{2} - c.$$

Then:

$$f(\mathbf{x}, \hat{y}) = \begin{cases} \hat{y}, & \text{w.p. } p, \\ 1/2, & \text{w.p. } 1-p. \end{cases}$$

# Bias-Aware Mechanism (BAM)

## Mechanism: Bias-Aware Mechanism (BAM)

Given prediction $\hat{y}$, compute

$$c = \left|\hat{y} - \frac{1}{2}\right|, \quad p = \frac{1}{2} - c.$$

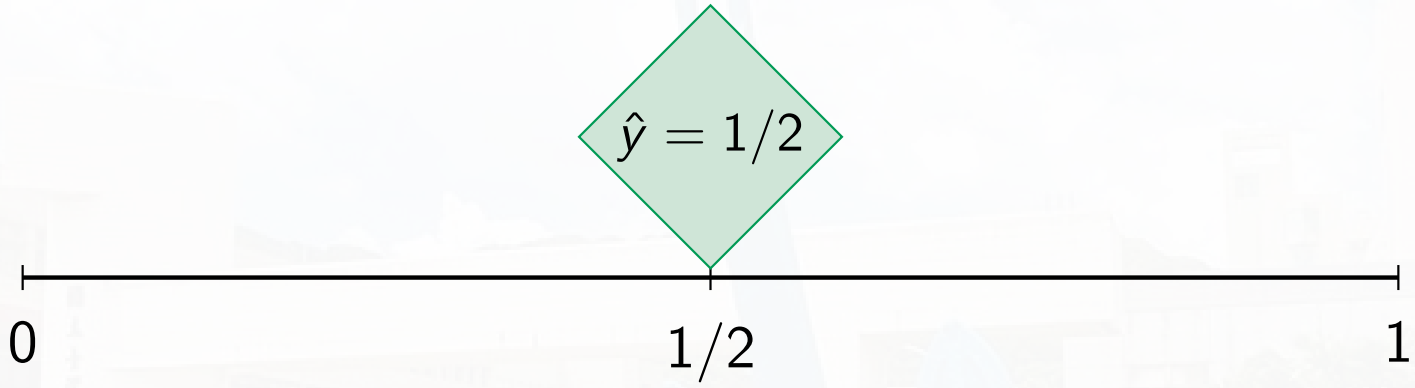
Then:

$$f(\mathbf{x}, \hat{y}) = \begin{cases} \hat{y}, & \text{w.p. } p, \\ 1/2, & \text{w.p. } 1 - p. \end{cases}$$

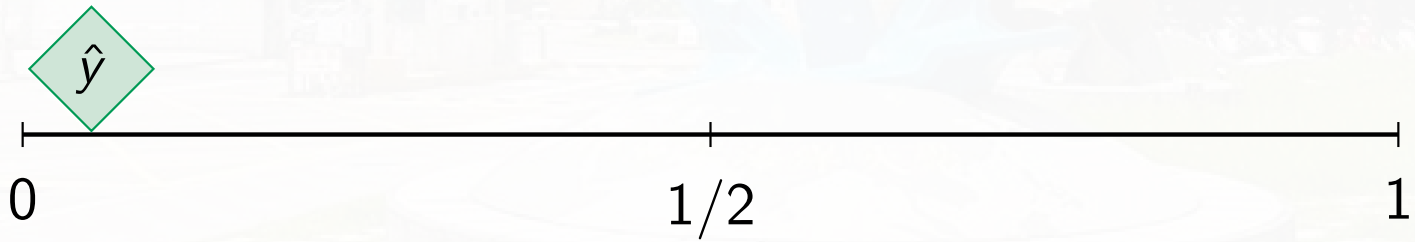
Again, the output distribution is independent of reported locations, so BAM is strategyproof.

Go to LRM

# BAM Probability Schedule




$p = 1/2$: prediction has maximal trust




$p$ close to 0: prediction has little trust

# BAM Guarantees

**Theorem 4.5**

BAM is anonymous and strategyproof. Its guarantees are:

| bias $c$ | consistency | robustness |
|---|---|---|
| $0 \leq c < 1/4$ | $7/4$ | $9/4$ |
| $1/4 \leq c \leq 1/2$ | $-4c^2 + 2$ | $c + 2$ |

- If the prediction is central, BAM safely uses it with some probability.
- If the prediction is extreme, BAM mostly falls back to $1/2$.

# Robustness Intuition for BAM

Assume $\hat{y} \leq 1/2$.

# Robustness Intuition for BAM

Assume $\hat{y} \leq 1/2$. $\Rightarrow$ $c = \frac{1}{2} - \hat{y} \Rightarrow \hat{y} = \frac{1}{2} - c = p$.

# Robustness Intuition for BAM

Assume $\hat{y} \leq 1/2$. $\Rightarrow$ $c = \frac{1}{2} - \hat{y} \Rightarrow \hat{y} = \frac{1}{2} - c = p$.

- Probability of using prediction: $p = \hat{y}$.
- Probability of using center: $1 - p = 1 - \hat{y}$.
- Any agent's utility from $\hat{y}$ is at least $\hat{y}$.
- Any agent's utility from $1/2$ is at least $1/2$.

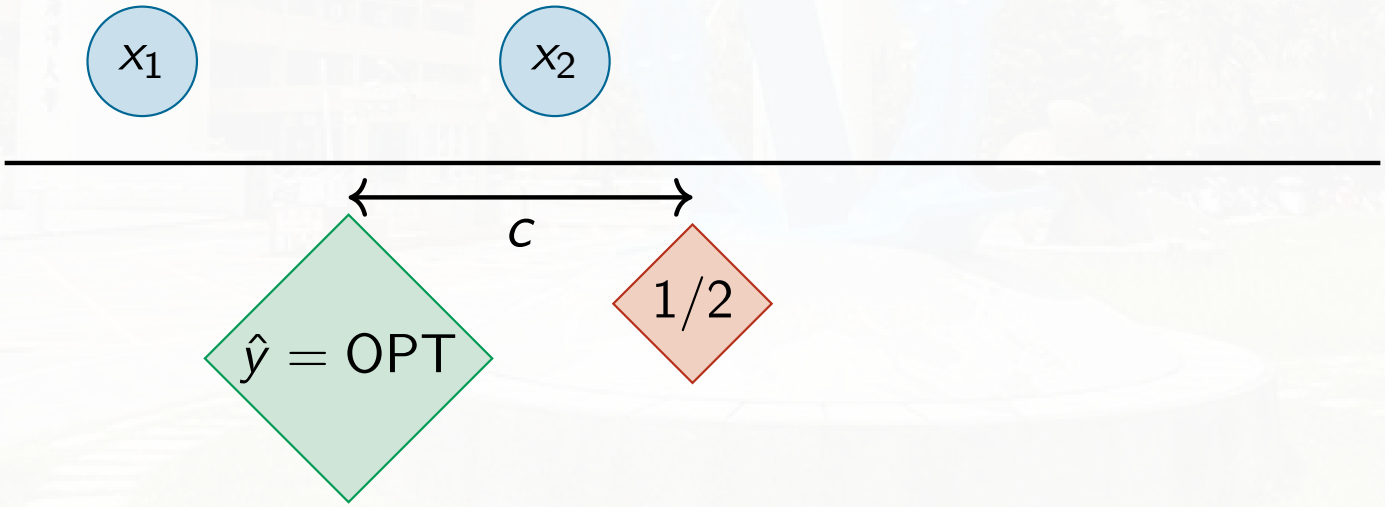
**Worst case**

A bad case places one agent near the chosen safe center and another near the far endpoint.

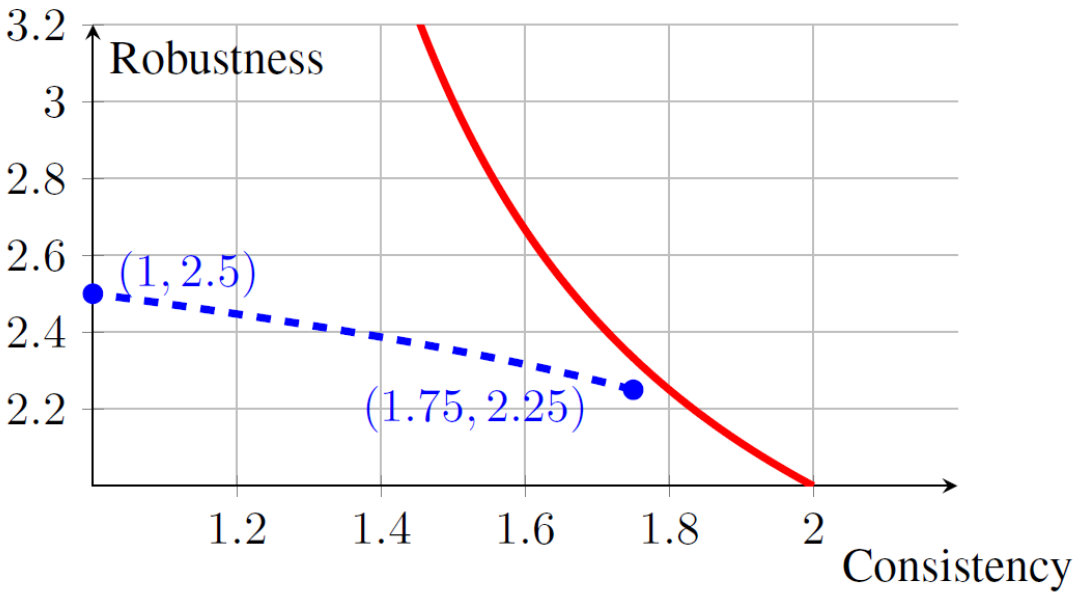
# Consistency Intuition for BAM

If $\hat{y} = \text{OPT}(\mathbf{x})$, then placing the facility at $\hat{y}$ is optimal.

- With probability $p$, BAM uses the optimal location.
- With probability $1 - p$, it uses the center $1/2$.
- The loss depends on how far the optimal location is from $1/2$.



# BAM vs. $\alpha$-BIM




BAM gives a better consistency–robustness profile than the deterministic frontier $\alpha$-BIM.

# Why Not Combine BIM with LRM?

A tempting idea:

- use $\widehat{y}$ inside the BIM interval;
- use the LRM mechanism outside the interval.

**Paper's conclusion**

This hybrid is worse than the original $\alpha$-BIM in its guarantees.

- Randomization helps in the no-prediction setting.
- But blindly inserting it into a prediction framework can harm consistency and robustness.

# Proof of Theorem 4.5: Reduction

- By Lemma 3.1, it suffices to consider two-agent instances $\mathbf{x} = (x_1, x_2),\ x_1 < x_2$.
- By symmetry, assume $\hat{y} \leq \frac{1}{2}$. Then $c = \frac{1}{2} - \hat{y}, \quad p = \frac{1}{2} - c = \hat{y}$.
- So BAM becomes:

$$\begin{cases} \text{output } \hat{y}, & \text{with probability } \hat{y}, \\ \text{output } \frac{1}{2}, & \text{with probability } 1 - \hat{y}. \end{cases}$$

- For a two-agent instance, the optimal envy ratio is

$$\mathrm{ER}(\mathrm{mid}(\mathbf{x}), \mathbf{x}) = 1.$$

- Thus, it suffices to upper-bound the envy ratio achieved by BAM.

# Robustness of BAM (1/2)

- Consider an arbitrary prediction $\hat{y} \leq \frac{1}{2}$.
  - For the output $\hat{y}$, every agent has utility at least $\hat{y}$.
  - For the output $\frac{1}{2}$, every agent has utility at least $\frac{1}{2}$.

# Robustness of BAM (1/2)

- Consider an arbitrary prediction $\hat{y} \leq \frac{1}{2}$.
  - For the output $\hat{y}$, every agent has utility at least $\hat{y}$.
  - For the output $\frac{1}{2}$, every agent has utility at least $\frac{1}{2}$.
- The worst case occurs when the right agent is at $x_2 = 1$.

# Robustness of BAM (1/2)

- Consider an arbitrary prediction $\hat{y} \leq \frac{1}{2}$.
  - For the output $\hat{y}$, every agent has utility at least $\hat{y}$.
  - For the output $\frac{1}{2}$, every agent has utility at least $\frac{1}{2}$.
- The worst case occurs when the right agent is at $x_2 = 1$.
- Moreover, it suffices to consider $x_1 \in \left[\hat{y}, \frac{1}{2}\right]$.

# Robustness of BAM (1/2)

- Consider an arbitrary prediction $\hat{y} \leq \frac{1}{2}$.
  - For the output $\hat{y}$, every agent has utility at least $\hat{y}$.
  - For the output $\frac{1}{2}$, every agent has utility at least $\frac{1}{2}$.
- The worst case occurs when the right agent is at $x_2 = 1$.
- Moreover, it suffices to consider $x_1 \in \left[\hat{y}, \frac{1}{2}\right]$.
  - After setting $x_2 = 1$, the two realized envy ratios are

$$\frac{1 - |x_1 - \hat{y}|}{\hat{y}} \quad \text{and} \quad \frac{1 - |x_1 - \frac{1}{2}|}{\frac{1}{2}}.$$

  - Moving $x_1 < \hat{y}$ to $\hat{y}$, or moving $x_1 > \frac{1}{2}$ to $\frac{1}{2}$, weakly increases both ratios. Hence, for the worst-case analysis, we may assume $x_1 \in \left[\hat{y}, \frac{1}{2}\right]$.

# Robustness of BAM (2/2)

- Then the envy ratio is bounded by

$$\mathrm{ER}(f(\mathbf{x},\hat{y}),\mathbf{x}) \leq \hat{y} \cdot \frac{1-(x_1-\hat{y})}{\hat{y}} + (1-\hat{y}) \cdot \frac{1-(\frac{1}{2}-x_1)}{\frac{1}{2}}.$$

- Simplifying, $\mathrm{ER}(f(\mathbf{x},\hat{y}),\mathbf{x}) \leq 2 + x_1(1-2\hat{y})$.
- Since $x_1 \leq \frac{1}{2}$, we have $\mathrm{ER}(f(\mathbf{x},\hat{y}),\mathbf{x}) \leq 2 + \frac{1}{2} - \hat{y} = 2 + c$.
- Thus BAM is $(2+c)$-robust.

# Robustness Bound by Ranges

- We have shown $\beta(c) \leq 2 + c$.
- Therefore, when $c \in \left[\frac{1}{4}, \frac{1}{2}\right]$, BAM is $(2 + c)$-robust.

# Robustness Bound by Ranges

- We have shown $\beta(c) \leq 2 + c$.
- Therefore, when $c \in \left[\frac{1}{4}, \frac{1}{2}\right]$, BAM is $(2 + c)$-robust.
- When $c \in \left[0, \frac{1}{4}\right)$, we have

$$2 + c < 2 + \frac{1}{4} = \frac{9}{4}.$$

- Hence BAM is $\frac{9}{4}$-robust.

# Consistency of BAM: Setup

- Now assume the prediction is accurate: $\hat{y} = \text{mid}(\mathbf{x})$.
- Let $\delta = \dfrac{x_2 - x_1}{2}$. Then $x_1 = \hat{y} - \delta$, $x_2 = \hat{y} + \delta$.
- Since $x_1 \geq 0$, we have $\delta \leq \hat{y}$.
- BAM outputs $\hat{y}$ w.p. $\hat{y}$, and $\frac{1}{2}$ w.p. $1 - \hat{y}$.
- If the facility is placed at $\hat{y} = \text{mid}(\mathbf{x})$, the envy ratio is 1.
- Thus only the outcome $\frac{1}{2}$ may create envy.

# Consistency: Case 1, $\hat{y} \leq \frac{1}{4}$

- Assume $\hat{y} \leq \frac{1}{4}$. Then $x_2 = \hat{y} + \delta \leq 2\hat{y} \leq \frac{1}{2}$.
- When the facility is placed at $\frac{1}{2}$, agent $x_2$ is closer to the facility than agent $x_1$. Hence

$$\mathrm{ER}\left(\frac{1}{2}, \mathbf{x}\right) = \frac{1 - (\frac{1}{2} - \hat{y} - \delta)}{1 - (\frac{1}{2} - \hat{y} + \delta)}.$$

- This expression is increasing in $\delta$, and $\delta \leq \hat{y}$, so $\mathrm{ER}\left(\frac{1}{2}, \mathbf{x}\right) \leq \frac{\frac{1}{2} + 2\hat{y}}{\frac{1}{2}} = 1 + 4\hat{y}$.
- Therefore, $\gamma \leq \hat{y} \cdot 1 + (1 - \hat{y})(1 + 4\hat{y}) = 1 + 4\hat{y} - 4\hat{y}^2$.
- Since $c = \frac{1}{2} - \hat{y}$, we have $1 + 4\hat{y} - 4\hat{y}^2 = 2 - 4c^2$.
- Thus, for $c \in [\frac{1}{4}, \frac{1}{2}]$, we have $\gamma \leq 2 - 4c^2$.

# Consistency: Case 2, $\hat{y} > \frac{1}{4}$ (1/2)

- Assume $\hat{y} > \frac{1}{4}$. Note that $\widehat{y} = \text{mid}(\mathbf{x})$ and $\widehat{y} - \delta = x_1$, $\widehat{y} + \delta = x_2$, $c < \frac{1}{4}$.
- When the facility is placed at $\frac{1}{2}$, the worst case occurs at $\delta = \hat{y}$ $(= \text{mid}(\mathbf{x}))$.

# Consistency: Case 2, $\hat{y} > \frac{1}{4}$ (1/2)

- Assume $\hat{y} > \frac{1}{4}$. Note that $\widehat{y} = \text{mid}(\mathbf{x})$ and $\widehat{y} - \delta = x_1$, $\widehat{y} + \delta = x_2$, $c < \frac{1}{4}$.
- When the facility is placed at $\frac{1}{2}$, the worst case occurs at $\delta = \hat{y}$ $(= \text{mid}(\mathbf{x}))$.
  - Let $c = \frac{1}{2} - \hat{y}$. At facility location $\frac{1}{2}$,

$$\text{ER}\left(\frac{1}{2}, \mathbf{x}\right) = \frac{1 - |\frac{1}{2} - x_2|}{1 - |\frac{1}{2} - x_1|} = \frac{1 - |c - \delta|}{1 - (\frac{1}{2} - \widehat{y} + \widehat{y} - x_1)} = \frac{1 - |c - \delta|}{1 - c - \delta}.$$

    This is nondecreasing in $\delta \in [0, \hat{y}]$. Since $\delta \leq \hat{y}$, its maximum occurs at

$$\delta = \hat{y}, \quad \left(\text{check ER}'\left(\frac{1}{2}, \mathbf{x}\right) \text{ of two cases: } |c - \delta| = c - \delta \text{ or } \delta - c\right)$$

    which corresponds to $\mathbf{x} = (0, 2\hat{y})$.
- Then, $\text{ER}\left(\frac{1}{2}, \mathbf{x}\right) \leq \frac{1 + c - \hat{y}}{1 - c - \hat{y}} \leq \frac{1 + \frac{1}{2} - 2\hat{y}}{\frac{1}{2}} = 3 - 4\hat{y}$.

# Consistency: Case 2, $\hat{y} > \frac{1}{4}$ (2/2)

- Therefore, $\gamma \leq \hat{y} \cdot 1 + (1 - \hat{y})(3 - 4\hat{y}) = 3 - 6\hat{y} + 4\hat{y}^2$.
- Using $c = \frac{1}{2} - \hat{y}$, we obtain that

$$3 - 6\hat{y} + 4\hat{y}^2 = 1 + 2c + 4c^2.$$

- For $c \in \left[0, \frac{1}{4}\right)$, this is maximized at $c = \frac{1}{4}$, so

$$1 + 2c + 4c^2 \leq 1 + \frac{1}{2} + \frac{1}{4} = \frac{7}{4}.$$

- Hence, for $c \in [0, \frac{1}{4})$, we have $\gamma \leq \frac{7}{4}$.

# Proof of Theorem 4.5: Conclusion

- Combining the two parts:

$$\beta \leq 2 + c \text{ and } \gamma \leq \begin{cases} 2 - 4c^2, & c \in [\frac{1}{4}, \frac{1}{2}], \\ \frac{7}{4}, & c \in [0, \frac{1}{4}). \end{cases}$$

- Therefore BAM satisfies

$$\begin{cases} (2 - 4c^2)\text{-consistency and } (2 + c)\text{-robustness}, & c \in \left[\frac{1}{4}, \frac{1}{2}\right], \\ \frac{7}{4}\text{-consistency and } \frac{9}{4}\text{-robustness}, & c \in \left[0, \frac{1}{4}\right). \end{cases}$$

- Thus Theorem 4.5 follows.

# Outline



# Summary of Mechanisms

| Mechanism | Prediction? | Main guarantee |
|---|---|---|
| Constant 1/2 | no | deterministic ratio 2 |
| $\alpha$-BIM | yes | $\alpha$-consistent, $\alpha/(\alpha-1)$-robust; deterministic optimal |
| LRM constant | no | randomized ratio $1+2/\sqrt{5} \approx 1.8944$ |
| BAM | yes | randomized; improves over $\alpha$-BIM trade-off |

# Open Directions

- Tight bounds for randomized mechanisms with predictions.
- Closed-form error-parameterized approximation for BAM.
- Extensions to other fairness objectives in facility location.
- Extensions beyond the one-dimensional interval.

# Main Referenced Works

[1] Haris Aziz, Yuhang Guo, Alexander Lam, and Houyu Zhou.
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# Discussions