Testing induced P_3-freeness

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Outline

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1. Introduction
2. Testing induced P_3-freeness
3. Concluding remarks

Introduction (model)

- Graph model: dense graph (adjacency matrix) for $G(V, E)$.
  - undirected, no self-loops, ≤ 1 edge between any $u, v \in V$
  - $|V| = n$ vertices and $|E| = \Omega(n^2)$ edges.

- A graph property:
  - A set of graphs closed under isomorphisms.

- Let $\mathbb{P}$ be a graph property.
  - ϵ-far from satisfying $\mathbb{P}$:
    - $\geq \epsilon n^2$ edges should be removed or added to let the graph satisfy $\mathbb{P}$

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- Property testing:
  - it does NOT precisely determine YES or NO for a decision problem;
  - requires sublinear running time

- A property tester for $\mathbb{P}$:
  - A randomized algorithm such that
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    - it answers "NO" with probability of $\geq 2/3$ if G is ϵ-far from satisfying $\mathbb{P}$.

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Surveys...

- E. Fischer: The art of uninformed decisions: A primer to property testing. *The Computational Complexity Column of The Bulletin of the European Association for Theoretical Computer Science*, **75** (2001), pp. 97–126.
- O. Goldreich: *Combinatorial property testing - a survey*. Randomization Methods in Algorithm Design (P. M. Pardalos, S. Rajasekaran and J. D. P. Rolim eds.), AMS-DIMACS (1998), pp. 45–60.
- D. Ron: Property testing. *Handbook of Randomized Computing*, Vol. II, Kluwer Academic Publishers (P. M. Pardalos, S. Rajasekaran and J. D. P. Rolim eds.), 2001, pp. 597–649.

Introduction (testing graph properties)

- Throughout this talk, we focus on graph properties and the dense graph model.
- A property tester has the ability to make queries and then make decision by making use of the answers of queries.
  - To see whether a desired pair of vertices are adjacent or not.
- And, we care about query complexities in this talk.
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- $\mathbb{P}$ is testable if
  - $\exists$ a property tester for $\mathbb{P}$ such that its query complexity is independent of n.
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Induced H-freeness

- $G[H]$: the induced subgraph of G on H.
- $\mathbb{P}^*_H$: the property that a graph having no H as an induced subgraph.
- A graph G satisfies $\mathbb{P}^*_H \Leftrightarrow G$ does not have H as an induced subgraph.

Goals of this talk

- We show that $\mathbb{P}^*_{P_3}$ is easily testable.
  - Only $O(\log^2(1/\epsilon)/\epsilon^2)$ queries are required.

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The property tester for $\mathbb{P}^*_{P_3}$

- The property tester is as follows:
  1. Pick a random subset of $10\log(1/\epsilon)/\epsilon$ vertices.
  2. Check if there is an induced copy of P_3 spanned by this set.
- The query complexity is at most $O(\log^2(1/\epsilon)/\epsilon^2)$.
- If G satisfies $\mathbb{P}^*_{P_3}$, the algorithm always answers correctly (i.e., answers YES since there is no induced P_3).
- We have to show that if G is ϵ-far from satisfying $\mathbb{P}^*_{P_3}$, the algorithm finds an induced copy of P_3 with probability $\geq 2/3$.

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High degree vertices

- Let *HIGH* be the set $\{v \in V(G) \mid \deg(v) \geq \frac{\epsilon n}{4}\}$.
  - Intuitively, vertices of HIGH have high contribution to G being ϵ-far from satisfying $\mathbb{P}^*_{P_3}$.

HIGH has high contribution indeed!

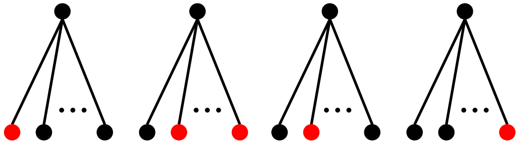
Claim 1

Assume that G is ϵ-far from satisfying $\mathbb{P}^*_{P_3}$ and $W \subseteq V(G)$ contains at least $|HIGH| - \frac{\epsilon}{4}n$ vertices of HIGH, then it requires to add or remove $\geq \frac{\epsilon}{2}n^2$ edges to make $G[H]$ satisfy $\mathbb{P}^*_{P_3}$.

Randomly chosen subset of vertices are Good w.h.p.

Definition 1

We call a set $A \subseteq V(G)$ Good if at least $|\mathrm{HIGH}| - \frac{\epsilon}{4}n$ vertices of HIGH have a neighbor in A.



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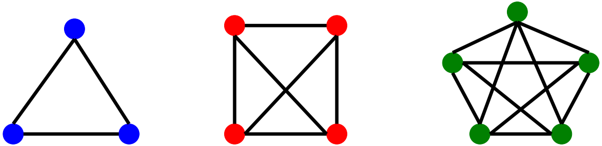
Claim 2

A randomly chosen subset $A \subseteq V(G)$ *of size* $8 \log(1/\epsilon)/\epsilon$ *is Good with probability at least* $7/8$.

A well-known observation for induced P_3-free graphs

- A graph is induced P_3-free if and only if it is disjoint union of cliques.



Correctness and query complexity of the algorithm

- First we choose a random subset $A \subset V$ of size $8 \log(1/\epsilon)/\epsilon$.
- Assume that A is Good (this is not true with probability $\leq 1/8$).
- If A contains an induced copy of P_3, then we are done.

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- Otherwise, (i.e., A contains no induced copy of P_3)
  - Let W be the set of all the vertices $v \in V$ that ≥ 1 neighbor in A.
  - Recall that G is assume to be ϵ-far from satisfying $\mathbb{P}^*_{P_3}$, and A is assumed to be Good.
- And of course, we can assume that A can be partitioned into disjoint union of cliques $C_1, C_2, \ldots, C_r$, for some integer r.

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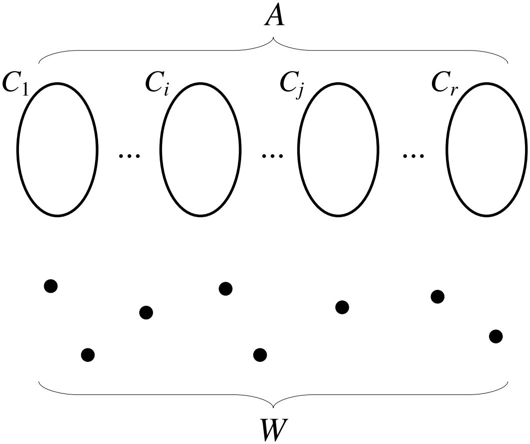
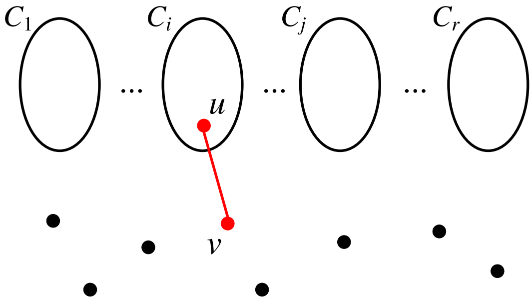

A
C1
Ci
Cj
Cr
W

- If a vertex $v \in W$ is connected to $u \in C_i \subseteq A$, it follows that if W can be partitioned into cliques $D_1, \ldots, D_r$, where for $1 \leq i \leq r$, $C_i \subseteq D_i$, then v would have to belong to D_i.



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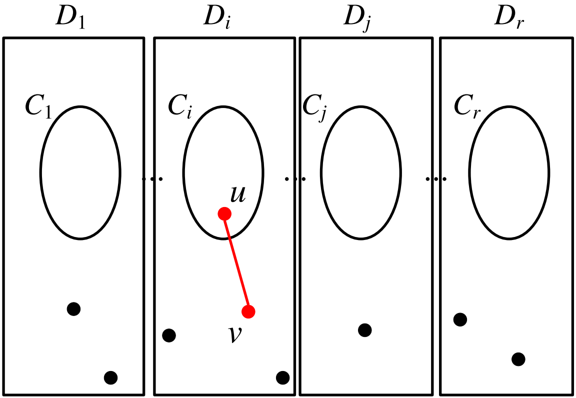
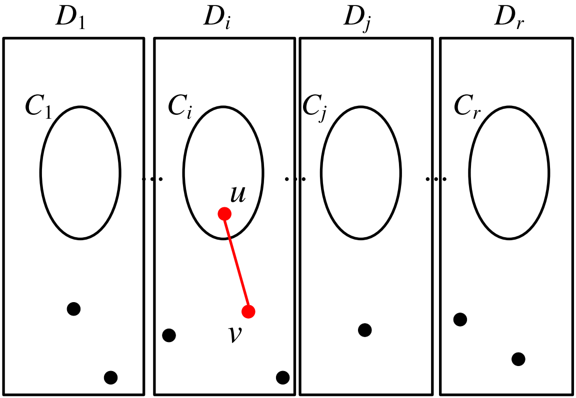
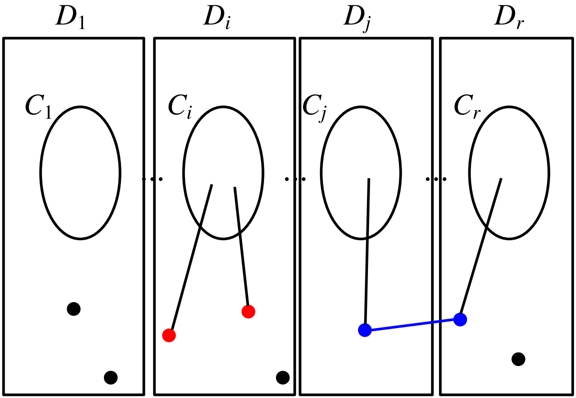


- For each $v \in W$ connected to $u \in C_i$, assign v the number i. If v is connected to vertices that belong to different C_i's, then assign v any of these numbers.
- The numbering induces a partition of W into r subsets.

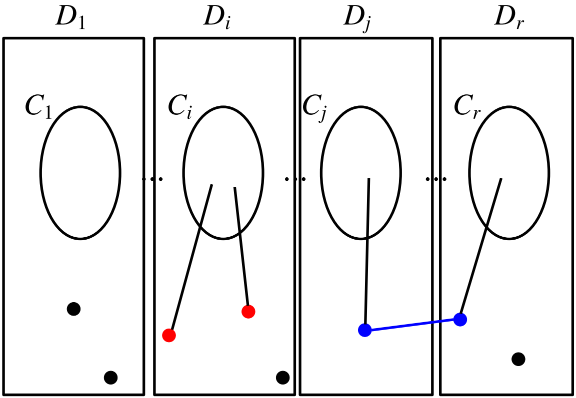


- Violating pairs: "$s, t \in D_i$ but s, t are not connected" or "$s \in D_i, t \in D_j$ for $i \neq j$ but s, t are connected".
- There are at least $\frac{\epsilon}{2}n^2$ violating pairs of vertices in W (for A is Good, so that W contains many vertices of HIGH).



- Therefore, choosing a set B of $8/\epsilon$ randomly chosen pairs of vertices fails to find violating pairs with probability of at most

$$\left(1 - \frac{\epsilon n^2/2}{n(n-1)/2}\right)^{8/\epsilon} < \left(1 - \frac{\epsilon}{2}\right)^{8/\epsilon} < e^{-4} < \frac{1}{8}.$$



To sum up

- By Claim 2, $\mathbf{Pr}[A \text{ is NOT Good}] \leq \frac{1}{8}$.
- $\mathbf{Pr}[B \text{ does NOT contain any violating pair of vertices}] \leq \frac{1}{8}$.
- Hence with probability $< \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$ the induced subgraph $G[A \cup B]$ is not induced P_3-free.
- Since $|A| + |B| = O(8 \log(1/\epsilon)/\epsilon + 8/\epsilon) = O(8 \log(1/\epsilon)/\epsilon)$, the proof is complete!

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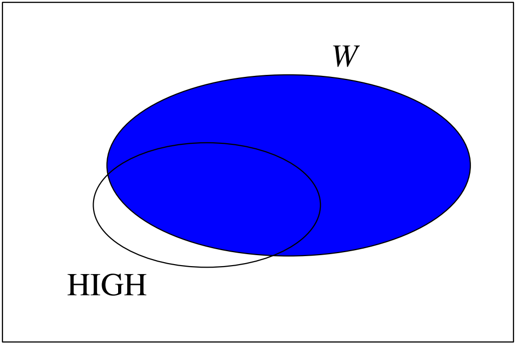
Back to the proofs of claims

Claim 1

Assume that G is ϵ-far from satisfying $\mathbb{P}^*_{P_3}$ and $W \subseteq V(G)$ contains at least $|\text{HIGH}| - \frac{\epsilon}{4}n$ vertices of HIGH, then it requires to add or remove $\geq \frac{\epsilon}{2}n^2$ edges to make $G[H]$ satisfy $\mathbb{P}^*_{P_3}$.

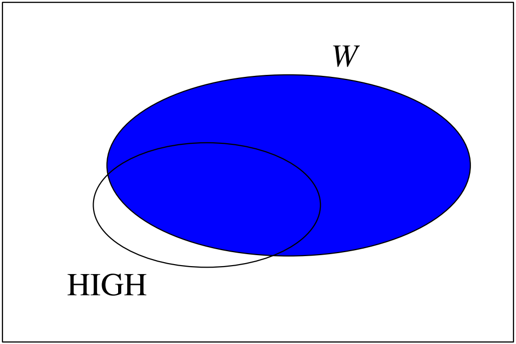
Proof of Claim 1

- Assume this is not the case (proof by contradiction).



Proof of Claim 1 (contd.)

- That is, we can make less than $\frac{\epsilon}{2}n^2$ changes (edge removals or edge additions) within W and get a graph that contains no induced copy of P_3 within W.



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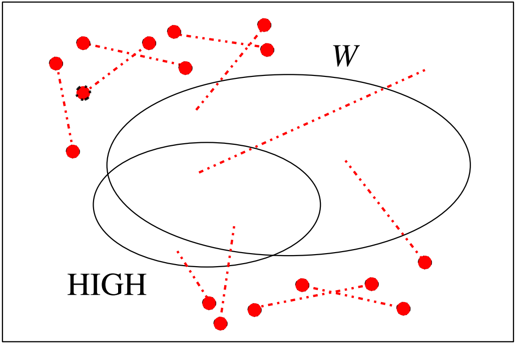
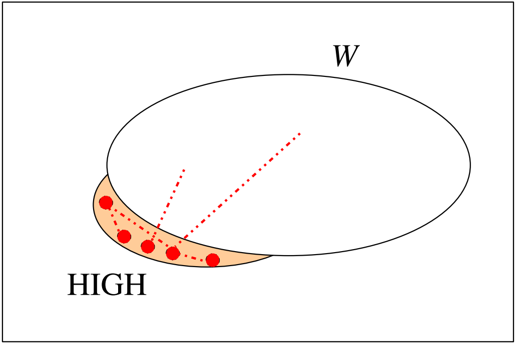
- Then we remove all the edges touching a vertex not in $W \cup \text{HIGH}$.
- $\leq n \cdot \frac{\epsilon}{4} n$ such edges.



Proof of Claim 1 (contd.)

- Then we remove any edge touching a vertex in HIGH $\setminus W$.
- $\leq \frac{\epsilon}{4}n \cdot n$ such edges since $|\text{HIGH} \setminus W| \leq \frac{\epsilon}{4}n$.



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- Thus we obtain a graph that satisfies $\mathbb{P}^*_{P_3}$.
- $< \epsilon n^2$ edges are added or removed in G, so the remaining graph is not ϵ-far from satisfying $\mathbb{P}^*_{P_3}$.
  - This contradicts the assumption!

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Claim 2

A randomly chosen subset $A \subseteq V(G)$ of size $8 \log(1/\epsilon)/\epsilon$ is Good with probability at least $7/8$.

Proof of Claim 2

- Let A be a randomly chosen subset of size $8\log(1/\epsilon)/\epsilon$.
- Consider a vertex $v \in$ HIGH.
- Since v has at least $\frac{\epsilon}{4}n$ neighbors, the probability that A does not contain any neighbor of v is at most

$$\left(1-\frac{\epsilon}{4}\right)^{8\log(1/\epsilon)/\epsilon} = \left[\left(1-\frac{\epsilon}{4}\right)^{\frac{-4}{\epsilon}}\right]^{-2\log(\frac{1}{\epsilon})} \leq e^{\log \epsilon^2} = \epsilon^2 \leq \frac{\epsilon}{32},$$

where we assume that $\epsilon < 1/32$.

▷ Exercise: Show that the above assumption can be loosed to $\epsilon < 1$ by letting $|A| = \frac{4\log(1/\epsilon)}{\epsilon} + \frac{20}{\epsilon}$.

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Proof of Claim 2 (contd.)

- We just obtained for $v \in \text{HIGH}$,
  $\mathbf{Pr}[A \text{ does not contain any neighbor of } v] \leq \frac{\epsilon}{32}$.
- Let X denote the number of vertices that belong to HIGH and have no neighbor in A.
- Since $|\text{HIGH}| \leq n$, we have $\mathbf{E}[X] \leq \frac{\epsilon}{32} \cdot n$ (by linearity of expectation).
- By Markov's inequality, $\mathbf{Pr}[X \geq \frac{\epsilon}{4} n] \leq \frac{\mathbf{E}[X]}{\frac{\epsilon}{4} n} \leq \frac{\epsilon n/32}{\epsilon n/4} = 1/8$.
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- Hence the proof is done.

Proof of Claim 2 (contd.)

- We just obtained for $v \in \text{HIGH}$,
  $\mathbf{Pr}[A \text{ does not contain any neighbor of } v] \leq \frac{\epsilon}{32}$.
- Let X denote the number of vertices that belong to HIGH and have no neighbor in A.
- Since $|\text{HIGH}| \leq n$, we have $\mathbf{E}[X] \leq \frac{\epsilon}{32} \cdot n$ (by linearity of expectation).
- By Markov's inequality, $\mathbf{Pr}[X \geq \frac{\epsilon}{4} n] \leq \frac{\mathbf{E}[X]}{\frac{\epsilon}{4} n} \leq \frac{\epsilon n/32}{\epsilon n/4} = 1/8$.
- Hence the proof is done.

Open problems

- Are P_4 and C_4 easily testable?

Thank you!