Query Complexity of Approximate Equilibria in Anonymous Games

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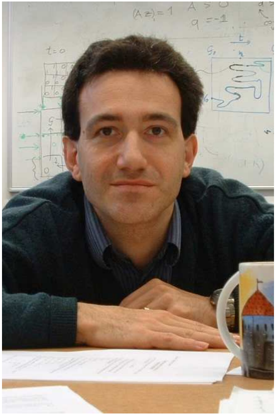


↑ Stefano Turchetta

← Paul W. Goldberg

NE, ϵ-NE, ϵ-WSNE

W \ M	O	F
O	$1, \frac{1}{2}$	$0, 0$
F	$0, 0$	$\frac{1}{2}, 1$

O: Opera;
F: Football Game

NE, ϵ-NE, ϵ-WSNE

$W \backslash M$	O	F
O	$1, \frac{1}{2}$	$0, 0$
F	$0, 0$	$\frac{1}{2}, 1$

Pure exact NE

NE, ϵ-NE, ϵ-WSNE

W \ M	O $\frac{1}{3}$	F $\frac{2}{3}$
O $\frac{2}{3}$	$1, \frac{1}{2}$	$0, 0$
F $\frac{1}{3}$	$0, 0$	$\frac{1}{2}, 1$

$$\mathbf{E}[u^W(\mathcal{X})] = \mathbf{E}[u^M(\mathcal{X})] = \frac{1}{3}.$$

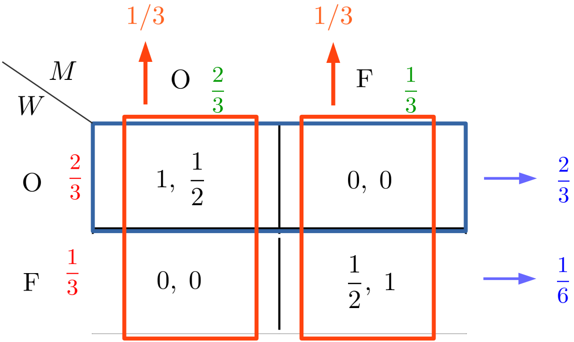
NE, ϵ-NE, ϵ-WSNE

$W \backslash M$	O $\frac{2}{3}$	F $\frac{1}{3}$
O $\frac{2}{3}$	$1, \frac{1}{2}$	$0, 0$
F $\frac{1}{3}$	$0, 0$	$\frac{1}{2}, 1$

$$\mathbf{E}[u^W(\mathcal{X})] = 1 \cdot \left(\frac{2}{3}\right)^2 + \frac{1}{2} \cdot \left(\frac{1}{3}\right)^2 = \frac{1}{2}.$$

$$\mathbf{E}[u^M(\mathcal{X})] = \frac{1}{2} \cdot \left(\frac{2}{3}\right)^2 + 1 \cdot \left(\frac{1}{3}\right)^2 = \frac{1}{3}.$$

NE, ϵ-NE, ϵ-WSNE



$$\mathbf{E}[u^W(\mathcal{X})] = 1 \cdot \left(\frac{2}{3}\right)^2 + \frac{1}{2} \cdot \left(\frac{1}{3}\right)^2 = \frac{1}{2}.$$

$$\mathbf{E}[u^M(\mathcal{X})] = \frac{1}{2} \cdot \left(\frac{2}{3}\right)^2 + 1 \cdot \left(\frac{1}{3}\right)^2 = \frac{1}{3}.$$

$\frac{1}{6}$ – NE

NE, ϵ-NE, ϵ-WSNE

W \ M	O $\frac{1}{2}$	F $\frac{1}{2}$
O $\frac{2}{3}$	1, 1	0, 0
F $\frac{1}{3}$	0, 0	1, 0

$$\mathbf{E}[u^W(\mathcal{X})] = 1 \cdot \frac{2}{3} \cdot \frac{1}{2} + 1 \cdot \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{2}.$$

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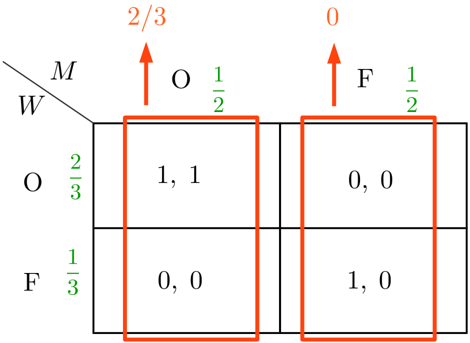
NE, ϵ-NE, ϵ-WSNE



$$\mathbf{E}[u^W(\mathcal{X})] = 1 \cdot \frac{2}{3} \cdot \frac{1}{2} + 1 \cdot \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{2}.$$

$$\mathbf{E}[u^M(\mathcal{X})] = 1 \cdot \frac{2}{3} \cdot \frac{1}{2} = \frac{1}{3}.$$

$\frac{1}{3}$-NE ✓

$\frac{1}{3}$-WSNE ✗

Outline

1. Introduction
   - Anonymous games
   - Query models
   - Related work
2. Exact NE
   - A lower bound
   - A game whose solution must have irrational numbers
   - Exact NE of symmetric anonymous games
   - Self-anonymous games
3. Main results: on two-strategy anonymous games
   - Two-strategy Lipschitz games
   - General two-strategy anonymous games
   - Lower bound*
4. The following work

Anonymous games

k-strategy anonymous games

$(n, k, \{u^i_j\}_{i\in[n], j\in[k]})$:

- n: # players;
- k: # pure strategies per player;
- $u^i_j : \prod^k_{n-1} \mapsto [0, 1]$: utility function
  - $\prod^k_{n-1} := \{(x_1, \ldots, x_k) \in ([k] \cup \{0\})^k \mid \sum_{j\in[k]} x_j = n - 1\}$.
  - ⋆ All possible ways to partition $n - 1$ players into the k strategies.

- Polynomial size representation.
  - $n \cdot k \cdot |\prod^k_{n-1}| = n \cdot k \cdot \binom{n-1+k-1}{k-1} = O(n^k)$.
  - The payoff to a player: does NOT depend on their identities.
  - Examples: *voting systems*, *traffic routing*, *auction settings*, ...

Anonymous games (contd.)

- $\mathcal{X}_i$: A mixed strategy of player i;
  - $\mathbf{E}[\mathcal{X}_i] = (p^i_1, \ldots, p^i_k)$.
    - p^i_j: the prob. that player i plays strategy j.
- $\mathcal{X}_{-i} := \sum_{\ell \in [n] \setminus \{i\}} \mathcal{X}_\ell$.
- $\mathbf{E}[u^i_j(\mathcal{X}_{-i})] := \sum_{x \in \prod^k_{n-1}} u^i_j(x) \cdot \Pr[\mathcal{X}_{-i} = x]$.
- Let $\mathcal{X} := (\mathcal{X}_i, \mathcal{X}_{-i})$. Then

$$\mathbf{E}[u^i(\mathcal{X})] := \sum_{j=1}^{k} p^i_j \cdot \mathbf{E}[u^i_j(\mathcal{X}_{-i})] \;\rightarrow\; \text{expected payoff of player } i$$

Two-strategy anonymous games

- $p_i := \mathbf{E}[X_i]$: a mixed strategy of player i.
  - X_i: Indicator r.v.; whether player i plays strategy 1.
- $X_{-i} := \sum_{\ell \in [n] \setminus \{i\}} X_\ell$.
- $\mathbf{E}[u^i_j(X_{-i})] := \sum_{x=0}^{n-1} u^i_j(x) \cdot \Pr[X_{-i} = x]$.
- Let $X := (X_i, X_{-i})$. Then

$$\mathbf{E}[u^i(X)] := \sum_{j=1}^{2} p^i_j \cdot \mathbf{E}[u^i_j(X_{-i})] \quad \rightarrow \quad \text{expected payoff of player } i$$

Concepts of equilibria

- $\mathcal{X}_i$ is a best-response iff $\mathbf{E}[u^i(\mathcal{X})] \geq \mathbf{E}[u^i_j(\mathcal{X}_{-i})]$.
- A Nash equilibrium (NE): requires each player to be best-responding to each other.
- $(\mathcal{X}_i)_{i \in [n]}$ is an
  - ϵ-approximate Nash equilibrium (ϵ-NE) if $\forall i \in [n], \forall j \in [k]$,
  $$\mathbf{E}[u^i(\mathcal{X})] + \epsilon \geq \mathbf{E}[u^i_j(\mathcal{X}_{-i})].$$
  - ϵ-approximate *well-supported* Nash equilibrium (ϵ-WSNE) if $\forall i \in [n], \forall j \in [k]$, and $\forall \ell \in \text{supp}(\mathbf{E}[\mathcal{X}_i])$,
  $$\mathbf{E}[u^i_\ell(\mathcal{X}_{-i})] + \epsilon \geq \mathbf{E}[u^i_j(\mathcal{X}_{-i})].$$

Sub-classes of anonymous games

Symmetric

$\forall i, \ell \in [n]$, $\forall j \in [k]$, $\forall x \in \prod_{n-1}^{k}$, that $u_j^i(x) = u_j^\ell(x)$ (sharing the same utility function).

Self-anonymous

$\forall i, \ell \in [n]$, $\forall j \in [k]$, $\forall x \in \{y \in \prod_{n-1}^{k} \mid y_\ell \neq 0\}$, that $u_j^i(x) = u_\ell^i(x + e_j - e_\ell)$ (one does NOT distinguish herself from the others).

Self-symmetric

symmetric + self-anonymous.

Lipschitz

Every player's utility function is Lipschitz continuous.

- $\forall i \in [n], j \in [k], \forall x, y \in \prod_{n-1}^{k}$, that $\left|u_j^i(x) - u_j^i(y)\right| \leq \lambda ||x - y||_1$.
- $\lambda \geq 0$: the Lipschitz constant.

Query models

- **Goal:** Finding equilibria while checking only a small fraction of the $O(n^k)$ payoffs of the game.

Single-payoff query

Given: $i \in [n]$, $j \in [k]$, and $x \in \prod_{n-1}^{k}$.
Return: $u_j^i(x)$.

- Query complexity of an algorithm: the expected # single-payoff queries it needs in the worst case.

Query models (contd.)

All-players query

Given: a pair (j, x) for $j \in [k]$, $x \in \prod_{n-1}^{k}$.
Return: $(u_j^1(x), u_j^2(x), \ldots, u_j^n(x))$.

Issues in complexities

- Computing an exact NE: **PPAD**-complete (normal-form) [Daskalakis, Goldberg, Papadimitriou @STOC'06].
  - BIMATRIX is **PPAD**-complete & does NOT have a FPTAS unless **PPAD** $\subseteq$ **P** [Chen, Deng, Teng @FOCS'06].
  - To find an ϵ-NE in an anonymous games with **7** strategies: **PPAD**-complete [Chen, Durfee, Orfanou @STOC'15].
- Yet, there exists a sub-exponential time algorithm to find an ϵ-NE (normal-form) [Lipton, Markakis, Mehta @EC'03].
- **OPEN:** Existence of a PTAS for these games.

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- **OPEN:** Existence of a PTAS for these games.

Issues in complexities (PTAS for anonymous games)

- Anonymous games admit a PTAS [Daskalakis & Papadimitriou @FOCS'07].
  - Currently best (two-strategy): $O(\text{poly}(n) \cdot (1/\epsilon)^{O(\log^2(1/\epsilon))})$, where $\text{poly}(n) \geq n^7$ [Daskalakis & Papadimitriou @STOC'09].
  - k-strategy anonymous games admit an $o(n^k)$ PTAS [Daskalakis, De, Tzamos + Diakonikolas, Kane, Stewart 2015].

Issues in complexities (query complexity)

- Exponential (deterministic/randomized) lower bounds for finding an ϵ-NE of n-player games.
- Such bounds do NOT hold in anonymous games.

Contribution of this paper

- *Exact NE*:
  - $\Omega(n^2)$ single-payoff queries for two-strategy anonymous games.
  - An example of anonymous games whose unique NE needs all players to randomize with an *irrational* amount of probability.
  - Tight query complexity bounds for finding *pure* NE in 2-strategy symmetric & k-strategy *self*-symmetric anonymous games.

Contribution of this paper (contd.)

- ϵ-NE:
  - "0" queries for finding an $O(n^{-1/2})$-WSNE for self-anonymous games.
  - 2-strategy anonymous game $G \mapsto$ 2-strategy self-anonymous game G'.
    - $\exists$ FPTAS for $G' \Rightarrow \exists$ FPTAS for G
  - A query-efficient algorithm that finds an ϵ-pure-NE in 2-strategy Lipschitz games (main subroutine).
  - A randomized PTAS for 2-strategy anonymous games.
    - For any $\epsilon \geq n^{-1/4}$, it finds an $O(\epsilon)$-NE with $\tilde{O}(n^{3/2})$ single-payoff queries and runs in time $\tilde{O}(n^{3/2})$.

On exact NE

Player R

x	0	1
$u_1^i(x)$	$\frac{1}{4}$	$\frac{3}{4}$
$u_2^i(x)$	$\frac{1}{2}$	$\frac{1}{2}$

Player C

x	0	1
$u_1^i(x)$	$\frac{3}{4}$	$\frac{1}{4}$
$u_2^i(x)$	$\frac{1}{2}$	$\frac{1}{2}$

R \ C	1	2
1	$\frac{3}{4}, \frac{1}{4}$	$\frac{1}{4}, \frac{1}{2}$
2	$\frac{1}{2}, \frac{3}{4}$	$\frac{1}{2}, \frac{1}{2}$

Player R

$u^i(x)$ \ x	0	1
$u_1^i(x)$	$\frac{1}{4}$	$\frac{3}{4}$
$u_2^i(x)$	$\frac{1}{2}$	$\frac{1}{2}$

Player C

$u^i(x)$ \ x	0	1
$u_1^i(x)$	$\frac{3}{4}$	$\frac{1}{4}$
$u_2^i(x)$	$\frac{1}{2}$	$\frac{1}{2}$

R \ C	$q = \frac{1}{2}$: 1	$1 - q = \frac{1}{2}$: 2
$p = \frac{1}{2}$: 1	$\frac{3}{4}, \frac{1}{4}$	$\frac{1}{4}, \frac{1}{2}$
$1 - p = \frac{1}{2}$: 2	$\frac{1}{2}, \frac{3}{4}$	$\frac{1}{2}, \frac{1}{2}$

No pure NE.

Player R

x	0	1
$u_1^i(x)$	$\frac{1}{4}$	$\frac{3}{4}$
$u_2^i(x)$	$\frac{1}{2}$	$\frac{1}{2}$

Player C

x	0	1
$u_1^i(x)$	$\frac{3}{4}$	$\frac{1}{4}$
$u_2^i(x)$	$\frac{1}{2}$	$\frac{1}{2}$

R \ C	1 ($q = \frac{1}{2}$)	2 ($1 - q = \frac{1}{2}$)
1 ($p = \frac{1}{2}$)	$\frac{3}{4}, \frac{1}{4}$	$\frac{1}{4}, \frac{1}{2}$
2 ($1 - p = \frac{1}{2}$)	$\frac{1}{2}, \frac{3}{4}$	$\frac{1}{2}, \frac{1}{2}$

adversary

Majority-minority game G

x	0	1	2	...	$n-2$	$n-1$
$u_1^i(x)$	$\frac{1}{2} - (\frac{1}{2} - \frac{1}{2n})$	$\frac{1}{2} - (\frac{1}{2} - \frac{3}{2n})$	$\frac{1}{2} - (\frac{1}{2} - \frac{5}{2n})$	...	$\frac{1}{2} + (\frac{1}{2} - \frac{3}{2n})$	$\frac{1}{2} + (\frac{1}{2} - \frac{1}{2n})$
$u_2^i(x)$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	...	$\frac{1}{2}$	$\frac{1}{2}$

(a) Payoff table for "majority-seeking" player i

x	0	1	2	...	$n-2$	$n-1$
$u_1^i(x)$	$\frac{1}{2} + (\frac{1}{2} - \frac{1}{2n})$	$\frac{1}{2} + (\frac{1}{2} - \frac{3}{2n})$	$\frac{1}{2} + (\frac{1}{2} - \frac{5}{2n})$	...	$\frac{1}{2} - (\frac{1}{2} - \frac{3}{2n})$	$\frac{1}{2} - (\frac{1}{2} - \frac{1}{2n})$
$u_2^i(x)$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	...	$\frac{1}{2}$	$\frac{1}{2}$

(b) Payoff table for "minority-seeking" player i

- $n/2$ majority-seeking players and $n/2$ minority-seeking players.
- x: (expected) # players other than i who play 1.

Majority-minority game G

x	0	1	2	...	$n-2$	$n-1$
$u_1^i(x)$	$\frac{1}{2}-(\frac{1}{2}-\frac{1}{2n})$	$\frac{1}{2}-(\frac{1}{2}-\frac{3}{2n})$	$\frac{1}{2}-(\frac{1}{2}-\frac{5}{2n})$	...	$\frac{1}{2}+(\frac{1}{2}-\frac{3}{2n})$	$\frac{1}{2}+(\frac{1}{2}-\frac{1}{2n})$
$u_2^i(x)$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	...	$\frac{1}{2}$	$\frac{1}{2}$

(a) Payoff table for "majority-seeking" player i

x	0	1	2	...	$n-2$	$n-1$
$u_1^i(x)$	$\frac{1}{2}+(\frac{1}{2}-\frac{1}{2n})$	$\frac{1}{2}+(\frac{1}{2}-\frac{3}{2n})$	$\frac{1}{2}+(\frac{1}{2}-\frac{5}{2n})$	...	$\frac{1}{2}-(\frac{1}{2}-\frac{3}{2n})$	$\frac{1}{2}-(\frac{1}{2}-\frac{1}{2n})$
$u_2^i(x)$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	...	$\frac{1}{2}$	$\frac{1}{2}$

(b) Payoff table for "minority-seeking" player i

- $P_1^i(s) := \mathbf{E}[u_1^i(x) - u_2^i(x)] = (\frac{x}{n} - \frac{1}{2} + \frac{1}{2n})$.
- The incentive for a majority player i' to play 1 is $\sum_{i \neq i'} \frac{p_i}{n} - \frac{n-1}{2n}$.
  - ▷ $\sum_{i \neq i'} p_i = \frac{n-1}{2}$.

Majority-minority game G

x	0	1	2	...	$n-2$	$n-1$
$u_1^i(x)$	$\frac{1}{2} - (\frac{1}{2} - \frac{1}{2n})$	$\frac{1}{2} - (\frac{1}{2} - \frac{3}{2n})$	$\frac{1}{2} - (\frac{1}{2} - \frac{5}{2n})$	...	$\frac{1}{2} + (\frac{1}{2} - \frac{3}{2n})$	$\frac{1}{2} + (\frac{1}{2} - \frac{1}{2n})$
$u_2^i(x)$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	...	$\frac{1}{2}$	$\frac{1}{2}$

(a) Payoff table for "majority-seeking" player i

x	0	1	2	...	$n-2$	$n-1$
$u_1^i(x)$	$\frac{1}{2} + (\frac{1}{2} - \frac{1}{2n})$	$\frac{1}{2} + (\frac{1}{2} - \frac{3}{2n})$	$\frac{1}{2} + (\frac{1}{2} - \frac{5}{2n})$	...	$\frac{1}{2} - (\frac{1}{2} - \frac{3}{2n})$	$\frac{1}{2} - (\frac{1}{2} - \frac{1}{2n})$
$u_2^i(x)$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	...	$\frac{1}{2}$	$\frac{1}{2}$

(b) Payoff table for "minority-seeking" player i

- Suppose that a majority player i' mixes with $p_{i'} \in (0, 1)$.
- $\mathbf{E}[\#\text{players of strategy } 1 - \#\text{players of strategy } 2] < 1$.
- No majority player may use pure strategy, otherwise he would deviate to the opposite one.
- Hence all majority players must MIX (the case for minority players is similar).

Majority-minority game G

x	0	1	2	$\dots$	$n-2$	$n-1$
$u_1^i(x)$	$\frac{1}{2} - (\frac{1}{2} - \frac{1}{2n})$	$\frac{1}{2} - (\frac{1}{2} - \frac{3}{2n})$	$\frac{1}{2} - (\frac{1}{2} - \frac{5}{2n})$	$\dots$	$\frac{1}{2} + (\frac{1}{2} - \frac{3}{2n})$	$\frac{1}{2} + (\frac{1}{2} - \frac{1}{2n})$
$u_2^i(x)$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\dots$	$\frac{1}{2}$	$\frac{1}{2}$

(a) Payoff table for "majority-seeking" player i

x	0	1	2	$\dots$	$n-2$	$n-1$
$u_1^i(x)$	$\frac{1}{2} + (\frac{1}{2} - \frac{1}{2n})$	$\frac{1}{2} + (\frac{1}{2} - \frac{3}{2n})$	$\frac{1}{2} + (\frac{1}{2} - \frac{5}{2n})$	$\dots$	$\frac{1}{2} - (\frac{1}{2} - \frac{3}{2n})$	$\frac{1}{2} - (\frac{1}{2} - \frac{1}{2n})$
$u_2^i(x)$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\dots$	$\frac{1}{2}$	$\frac{1}{2}$

(b) Payoff table for "minority-seeking" player i

- Suppose all players play pure strategies.
- If strategy 1 and 2 are both played by the same number of players, then all majority players will want to switch.
- If not, say strategy 1 is used by $> n/2$ players, it will be being used by a *minority* player who will want to switch!

An example of having unique & fully-mixed NE with irrationals

$$p_r = \frac{1}{12}(\sqrt{241} - 7),\ p_c = \frac{1}{16}(\sqrt{241} - 7),\ p_m = \frac{1}{36}(23 - \sqrt{241}).$$

1

	1	2
1	$(1, 0, 1)$	$(1, \frac{1}{2}, 0)$
2	$(0, 0, 0)$	$(\frac{1}{2}, \frac{1}{4}, 0)$

2

	1	2
1	$(1, 0, 0)$	$(0, \frac{1}{4}, \frac{1}{2})$
2	$(\frac{1}{2}, 1, \frac{1}{2})$	$(1, 0, 1)$

x	0	1	2
$u_1^r(x)$	0	1	1
$u_2^r(x)$	1	$\frac{1}{2}$	0

(a) r's payoff table

x	0	1	2
$u_1^c(x)$	1	0	0
$u_2^c(x)$	0	$\frac{1}{4}$	$\frac{1}{2}$

(b) c's payoff table

x	0	1	2
$u_1^m(x)$	0	0	1
$u_2^m(x)$	1	$\frac{1}{2}$	0

(c) m's payoff table

On symmetric anonymous games

Proposition 4.1

A pure NE of any 2-strategy n-player symmetric anonymous game can be found with $O(\log n)$ single-payoff queries.

- **Note:** Every 2-strategy symmetric game has a pure NE [Cheng *et al.* 2004].

Algorithm 1: SymmetricPNE

```
Data: The number of players n.
Result: The number of players m playing strategy 1 in a PNE.
begin
    return search(0, n − 1)
end
Procedure search(α, β)
    m := ⌊(α+β)/2⌋
    if m = α ∨ m = β then
        return m
    end
    Use queries to identify: u_1(m − 1), u_2(m − 1), u_1(m), u_2(m)
    if u_1(m − 1) ≥ u_2(m − 1) and u_1(m) ≤ u_2(m) then
        return m
    end
    if u_1(m − 1) < u_2(m − 1) then
        β := m
    else
        α := m
    end
    return search(α, β)
```

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Data: The number of players n.
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```
begin
    return search(0, n − 1)
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Procedure search(α, β)
    m := ⌊(α+β)/2⌋
    if m = α ∨ m = β then
        return m
    end
    Use queries to identify: u1(m − 1), u2(m − 1), u1(m), u2(m)
    if u1(m − 1) ≥ u2(m − 1) and u1(m) ≤ u2(m) then
        return m
    end
    if u1(m − 1) < u2(m − 1) then
        β := m
    else
        α := m
    end
    return search(α, β)
```

Check if $u_1(0) \leq u_2(0)$ or $u_1(n-1) \geq u_2(n-1)$ first!

Sketch of the proof of Proposition 4.1

Prove it by induction on k:

If a pure NE is in the search space of the k-th round and not yet bound, then there is a pure NE in the search space of round $k+1$.

- The base is trivial since the search space is $\{0, \ldots, n-1\}$.
- Suppose that after k-th round a pure NE is still in the search space but not found yet.
    - Let $\{\alpha_k, \ldots, \beta_k\}$ be the search space at step k, $m_k := \lfloor(\alpha + \beta)/2\rfloor$.
    - ⋆ By construction, $u_1(\alpha_k) > u_2(\alpha_k)$ & $u_1(\beta_k - 1) < u_2(\beta_k - 1)$.
- Case "$u_1(m_k - 1) < u_2(m_k - 1)$":
    - The search space: $\{\alpha_k, \ldots, m_k\}$.
    - Check a pure NE is at an $x \leq m_k - 1$.
        - Such an x must exist.
- Case "$u_1(m_k) \geq u_2(m_k)$": similarly holds.

Sketch of the proof of Proposition 4.1

Prove it by induction on k:

If a pure NE is in the search space of the k-th round and not yet bound, then there is a pure NE in the search space of round $k+1$.

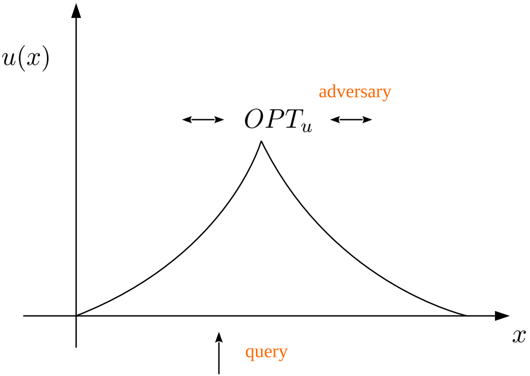
- The base is trivial since the search space is $\{0, \ldots, n-1\}$.
- Suppose that after k-th round a pure NE is still in the search space but not found yet.
  - Let $\{\alpha_k, \ldots, \beta_k\}$ be the search space at step k, $m_k := \lfloor(\alpha + \beta)/2\rfloor$.
  - ⋆ By construction, $u_1(\alpha_k) > u_2(\alpha_k)$ & $u_1(\beta_k - 1) < u_2(\beta_k - 1)$.
- Case "$u_1(m_k - 1) < u_2(m_k - 1)$":
  - The search space: $\{\alpha_k, \ldots, m_k\}$.
  - Check a pure NE is at an $x \leq m_k - 1$.
    - Such an x must exist.
- Case "$u_1(m_k) \geq u_2(m_k)$": similarly holds.

On symmetric anonymous games (contd.)

Proposition 4.2

Any algorithm that finds a pure NE in a 2-strategy n-player *self*-symmetric anonymous game needs $\Omega(\log n)$ single-payoff queries in the worst case.

- Such a game can be defined in terms of a utility function $u : \{0, \ldots, n\} \mapsto [0, 1]$.
- A pure NE corresponds to a local optimum of u.
- We restrict ourselves to functions u having a unique local optimum.


$u(x)$
adversary
OPT_u
x
query

On k-strategy self-symmetric anonymous games

- Every pure-strategy profile yields the same utility to all players.
- Such a game possesses a pure NE corresponding to a local optimum of u.

Lemma 4.1

For any constant k, the query complexity of search for a pure NE of k-strategy self-symmetric anonymous games, is within a constant factor of the query complexity of searching for a local optimum of the grid graph $[n]^{k-1}$.

Corollary 4.1

The randomized query complexity of searching for pure NE of self-symmetric anonymous games is $\Theta(n^{(k-1)/2})$ for constant $k \geq 5$.

Corollary 4.2

The deterministic query complexity of searching for pure NE of self-symmetric anonymous games is $\Theta(n^{k-2})$ for constant $k > 2$.

No query is required for self-anonymous games

Lemma 5.1

In any 2-strategy n player self-anonymous game, the mixed-strategy profile $s = (\frac{1}{2}, \dots, \frac{1}{2})$ is an $O(1/\sqrt{n})$-WSNE.

- Show that for any player $i \in [n]$,

$$|\mathbf{E}[u_1^i(X_{-i})] - \mathbf{E}[u_2^i(X_{-i})]| \leq \frac{e}{\pi} \cdot \frac{1}{\sqrt{n-1}}.$$

$$\sum_{x=0}^{n-1} (u_1^i(x) - u_2^i(x)) \cdot \Pr[X_{-i} = x]$$

$$= \sum_{x=0}^{n-2} (u_2^i(x+1) - u_2^i(x)) \cdot \Pr[X_{-i} = x] + (u_1^i(n-1) - u_2^i(n-1)) \cdot \Pr[X_{-1} = n-1]$$

$$= \sum_{x=0}^{n-2} (u_2^i(x+1) - u_2^i(x)) \cdot \binom{n-1}{x} \cdot \frac{1}{2^{n-1}} + (u_1^i(n-1) - u_2^i(n-1)) \cdot \frac{1}{2^{n-1}}$$

$$= \frac{1}{2^{n-1}} \left(\sum_{x=1}^{n-1} u_2^i(x) \cdot \left(\binom{n-1}{x-1} - \binom{n-1}{x} \right) + (u_1^i(n-1) - u_2^i(0)) \right)$$

Sketch of the proof of Lemma 5.1 (contd.)

- For all $x = 1, \ldots, (n-1)/2$, we have $\binom{n-1}{x-1} - \binom{n-1}{x} < 0$ and for all $x = (n-1)/2 + 1, \ldots, (n-1)$, we have $\binom{n-1}{x-1} - \binom{n-1}{x} > 0$.
- Let $u_2^i(x) = 0$ if $x \in \{1, \ldots, (n-1)/2\}$ and 1 otherwise.

$$\begin{aligned} & \sum_{x=0}^{n-1} (u_1^i(x) - u_2^i(x)) \cdot \Pr[X_{-i} = x] \\ \leq \quad & \frac{1}{2^{n-1}} \left(\sum_{x=\frac{n-1}{2}+1}^{n-1} \left(\binom{n-1}{x-1} - \binom{n-1}{x} \right) + 1 \right) \\ \ldots \\ = \quad & \frac{1}{2^{n-1}} \cdot \frac{(n-1)!}{((\frac{n-1}{2})!)^2}. \end{aligned}$$

- Using Stirling's bounds: $\sqrt{2\pi} \cdot n^{n+1/2} \cdot e^{-n} \leq n! \leq e \cdot n^{n+1/2} \cdot e^{-n}$, we obtain that the above is at most $\frac{e}{\pi} \cdot \frac{1}{\sqrt{n-1}}$.

No query is required for self-anonymous games (contd.)

Theorem 5.1

For constant k, in any k-strategy n-player self-anonymous game, letting every player randomized uniformly is an $O(1/\sqrt{n})$-WSNE.

Can be proved by induction (base case: $k = 2$).

- Suppose that it holds that in any $(k-1)$-strategy n-player self-anonymous game, every player $i \in [n]$ mixing uniformly is an $o(1/\sqrt{n})$-WSNE.

Proof of Theorem 5.1 (contd.)

- Let G_k be a k-strategy self-anonymous game.
- $X_i^{(\ell)}$: indicating whether player i plays strategy ℓ.
- $X_{-i}^{(k)} := \sum_{j \neq x} X_j^{(k)}$, # players other than i playing k in G_k.
- Observe that $\mathbf{E}[X_{-i}^{(k)}] = \frac{n-1}{k}$.
  - $\Pr\left[X_{-i}^{(k)} \geq \frac{2}{k}(n-1)\right] \leq e^{-\frac{n-1}{2k^2}}$ (Chernoff bound).

Proof of Theorem 5.1 (contd.)

$$\sum_{x_1+\ldots+x_k=n} (u_1^i(x_1,\ldots,x_k) - u_2^i(x_1,\ldots,x_k)) \cdot \Pr[X_{-i}^{(1)} = x_1,\ldots,X_{-i}^{(k)} = x_k]$$

$$= \sum_{x_k=0}^{n} \Pr[X_{-i}^{(k)} = x_k] \cdot$$

$$\sum_{x_1+\ldots+x_{k-1}=n-x_k} (u_1^i(x_1,\ldots,x_k) - u_2^i(x_1,\ldots,x_k)) \cdot \Pr[X_{-i}^{(1)} = x_1,\ldots,X_{-i}^{(k-1)} = x_{k-1} \mid X_{-i}^{(k)} = x_k]$$

$$= \sum_{x_k=0}^{n} \Pr[X_{-i}^{(k)} = x_k] \cdot O\left(\frac{1}{\sqrt{n-x_k}}\right)$$

$$\leq \sum_{x_k=0}^{\frac{2}{k}(n-1)} \Pr[X_{-i}^{(k)} = x_k] \cdot O\left(\frac{1}{\sqrt{n-2n/k}}\right) + \sum_{x_k=\frac{2}{k}(n-1)+1}^{n-1} e^{-\frac{n-1}{3k^2}} \cdot 1$$

$$\leq O\left(\sqrt{\frac{k}{kn-2}}\right) + \frac{k-2}{k} n \cdot e^{-\frac{n-1}{3k^2}} = O\left(\frac{1}{\sqrt{n}}\right).$$

Main results:

On finding ϵ-NE of 2-strategy anonymous games

δ-accurate all-players query

Let $(j, x) \in \{1, 2\} \times \{0, 1, \ldots, n-1\}$ be the input for an all-players query. For $\delta \geq 0$, a δ-accurate all-players query returns a tuple of values $(f_j^1(x), \ldots, f_j^n(x))$ such that for all $i \in [n]$, $|u_j^i(x) - f_j^i(x)| \leq \delta$.

- Within an additive error δ of the correct payoffs $(u_j^1(x), \ldots, u_j^n(x))$.

The algorithm: Using binary search!

Algorithm 2: Approximate NE Lipschitz

Data: δ-accurate query access to utility function $\bar{u}$ of n-player λ-Lipschitz game $\bar{G}$.
Result: pure-strategy $3(\delta + \lambda)$-NE of $\bar{G}$.
begin

Let $BR_1(i)$ be the number of players whose best response (as derived from the δ-accurate queries) is 1 when i of the other players play 1 and $n - 1 - i$ of the other players play 2.
Define $\phi(i) = BR_1(i) - i$. `// by construction,` $\phi(0) \geq 0$ `// and` $\phi(n-1) \leq 0$

If $BR_1(0) = 0$, **return** all-2's profile.
If $BR_1(n-1) = n$, **return** all-1's profile.
Otherwise, `// In this case,` $\phi(0) > 0$ `and` $\phi(n-1) \leq 0$
Find, via binary search, x such that $\phi(x) > 0$ and $\phi(x+1) \leq 0$.
Construct pure profile $\bar{p}$ as follows:
For each player i, if $\bar{u}_1^i(x) - \bar{u}_2^i(x) > 2\delta$, let i play 1, and if $\bar{u}_2^i(x) - \bar{u}_1^i(x) > 2\delta$, let i play 2. (The $\bar{u}_j^i$'s are δ-accurate.) Remaining players are allocated either 1 or 2, subject to the constraint that x or $x + 1$ players in total play 1.
return $\bar{p}$.

end

- **Note:** *Pure* ϵ-NE exists in Lipschitz games [Azrieli & Shmaya 2013].

Theorem 6.1

Let $\bar{G} = \left(n, 2, \{\bar{u}^i_j\}_{i\in[n], j\in\{1,2\}}\right)$ be an n-player, 2-strategy λ-Lipschitz anonymous game. Algorithm 2 finds a pure $3(\lambda+\delta)$-WSNE using $4\log n$ δ-accurate all-players queries.

- Suppose that the output $\bar{p}$ could not be constructed as the way described in the algorithm.
  - Suppose: $> x+1$ players are required to play 1 due to satisfying $\bar{u}^i_1(x) - \bar{u}^i_2(x) > 2\delta$.
- $\phi(x+1) = BR_1(x+1) - (x+1) \le 0 \Rightarrow n-(x+1)$ players whose payoffs to play 2 are at most 2δ less than their payoffs to play 1 (when $x+1$ other players play 1).
- When they play 2, they are 2δ-best-responding if $x+1$ players play 1.
  Lipschitz condition $\Rightarrow 2(\lambda+\delta)$-best-responding if x players play 1.
- So there is in fact a solution with only $x+1$ players playing 1.

Overview of the approach for general 2-strategy anonymous games

- We have a 2-strategy n-player anonymous game G.
- "Smooth" every player's utility function s.t. $G \to \lambda$-Lipschitz $\bar{G}$, for some λ.
  - The payoff received in $\bar{G}$ by player i when x other players playing 1 is given by the expected payoff received in G by player i when
    - x other players play 1 w.p. $1 - \zeta$ &
    - $n - 1 - x$ other players play 1 w.p. ζ.

  ⋆ $\bar{G}$: an $O\left(\frac{1}{\zeta\sqrt{n}}\right)$-Lipschitz game.
- Dealing with $\bar{G}$ by Algorithm 2.
  - But we are NOT allowed to query $\bar{G}$ directly...
  - Simulating each query to $\bar{G}$ with a small number of queries to G.
- Mapping the found approx. pure NE of $\bar{G}$ to G by
  - the players who play 1 in $\bar{G}$ → play 1 in G w.p. $1 - \zeta$.
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η-smoothed version of an anonymous game

Let $G = \left(n, 2, \{\bar{u}^i_j\}_{i\in[n], j\in\{1,2\}}\right)$ be an anonymous game. For $\eta > 0$, the η-smoothes version of G is a game $\bar{G} = (n, 2, \{\bar{u}^i_j\}_{i\in[n], j\in\{1,2\}})$ defined as follows.

Let $X^{(x)}_{-i} := \sum_{\ell\neq i} X_\ell$ denote the sum of $n-1$ Bernoulli random variables:

- x of them have expectation equal to $(1-\eta)$, and
- the remaining ones have expectation equal to η.

The payoff $\bar{u}^i_j(x)$ obtained by player $i \in [n]$ for playing strategy $j \in \{1,2\}$ against $x \in \{0, 1, \ldots, n-1\}$ is

$$\bar{u}^i_j(x) := \sum_{y=0}^{n-1} u^i_j(y) \cdot \Pr[X^{(x)}_{-i} = y] = \mathbf{E}[u^i_j(X^{(x)}_{-i})].$$

The algorithm

Algorithm 3: Approximate NE general payoffs

Data: ϵ; query access to utility function u of n-player anonymous game G; parameters τ (failure probability), δ (accuracy of queries).

Result: $O(\epsilon)$-NE of G.

begin

Set $\zeta = \epsilon$. Let $\bar{G}$ be the ζ-smoothed version of G, as in Definition 7.1.

`// By Lemma 7.1 and Lemma 7.5 it follows that`

`//` $\bar{G}$ `is` λ`-Lipschitz for` $\lambda = O(1/\zeta\sqrt{n})$.

Apply Algorithm 2 to $\bar{G}$, simulating each all-players δ-accurate query to $\bar{G}$ using multiple queries according to Lemma 7.2.

Let $\bar{p}$ be the obtained pure profile solution to $\bar{G}$.

Construct p by replacing probabilities of 0 in $\bar{p}$ with ζ and probabilities of 1 with $1 - \zeta$.

return p.

end

Lemma 7.2 (Simulation of a query to $\bar{G}$)

Let $\delta, \tau > 0$. Let X be the sum of $n-1$ Bernoulli random variables representing a mixed profile of $n-1$ players in an 2-strategy n-player anonymous game G.

Suppose we want to estimate, with additive error δ, the expected payoffs $\mathbf{E}[u^i_j(X)]$ for all $i \in [n]$, $j \in \{1,2\}$. This can be done w.p. $\geq 1-\tau$ using $(1/2\delta^2) \cdot \log(4n/\tau)$ all-players queries.

- Draw a set of N random samples $\{Z_1, \ldots, Z_N\}$ from X.
  - Compute each Z_i as a sum of 0/1 outcomes of $n-1$ biased coin flips.
  - For each Z_ℓ and each $j \in \{1,2\}$, make an all-players query telling us
    - $u^i_j(Z_\ell)$, for each i.
  - Total of $2N$ queries are made.

Proof of Lemma 7.2 (contd.)

- Let $\hat{U}_j^i := \frac{1}{N} \cdot \sum_{\ell=1}^{N} u_j^i(Z_\ell)$ (our estimate).
  - $\mathbf{E}[\hat{U}_j^i] = \mathbf{E}[u_j^i(X)]$.
- Using Hoeffding's inequality,

$$\Pr\left[\left|\hat{U}_j^i - \mathbf{E}[u_j^i(X)]\right| \geq \delta\right] \leq 2e^{-2\delta^2 N}$$

- We need that $2e^{-2\delta^2 N} \leq \tau/2n$.

  $\because$ $2n$ to estimate; all i & each $j \in \{1, 2\}$.

Lemma 7.1 [Daskalakis & Papadimitriou @J. Econom. Theory 2015]

X, Y: two random variables over $\{0, \ldots, n\}$ such that

$$||X - Y||_{TV} = \frac{1}{2} \cdot \sum_{x=0}^{n} |\Pr[X = x] - \Pr[Y = x]| \leq \delta.$$

Then, for $f : \{0, \ldots, n\} \mapsto [0, 1]$,

$$\sum_{x=0}^{n} f(x) \cdot (\Pr[X = x] - \Pr[Y = x]) \leq 2\delta.$$

Lemma 7.5

Let $X^{(j,n)} := \sum_{i\in[n]} X_i$ be the sum of n independent 0/1 random variables where

- $\mathbf{E}[X_i] = 1 - \zeta$ for all $i \leq j$;
- $\mathbf{E}[X_i] = \zeta$ for all $i > j$.

Then, for all $j \in [n]$, we have

$$\left\| X^{(j-1,n)} - X^{(j,n)} \right\|_{TV} = O\left(\frac{1}{\zeta\sqrt{n}}\right).$$

- Hence, $$\left\| X^{(x-1,n)} - X^{(x,n)} \right\|_{TV} = O\left(\frac{1}{\zeta\sqrt{n}}\right)$$

$$\Rightarrow \quad |\bar{u}^i_j(x-1) - \bar{u}^i_j(x)| = \left| \sum_{y=0}^{n-1} u^i_j(y) \cdot \Pr[X^{(x-1)}_{-i} = y] - \sum_{y=0}^{n-1} u^i_j(y) \cdot \Pr[X^{(x)}_{-i} = y] \right|$$

$$= \quad u^i_j(y) \cdot \left| \sum_{y=0}^{n-1} \Pr[X^{(x-1)}_{-i} = y] - \Pr[X^{(x)}_{-i} = y] \right| \leq O\left(\frac{1}{\zeta\sqrt{n}}\right)$$

for all $x \in [n-1]$ (by Lemma 7.1).

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- Hence,

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$$= \quad u^i_j(y)\cdot \left|\sum_{y=0}^{n-1} \Pr[X^{(x-1)}_{-i} = y] - \Pr[X^{(x)}_{-i} = y]\right| \le O\left(\frac{1}{\zeta\sqrt{n}}\right)$$

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Theorem 7.1

Let $G = \left(n, 2, \{\bar{u}^i_j\}_{i\in[n], j\in\{1,2\}}\right)$ be an anonymous game.

For $1/\epsilon = O(n^{-1/4})$, Algorithm 3 can be used to find (with prob. $\geq 3/4$) an ϵ-NE of G, using $O(n^{3/2} \cdot \log^2 n)$ single-payoff queries in time $O(n^{3/2} \cdot \log^2 n)$.

- Algorithm 2 finds a pure $O\left(\frac{1}{\zeta\sqrt{n}} + \delta\right)$-WSNE of $\bar{G}$.
- Simulate each δ-accurate query to $\bar{G}$ by $(1/2\delta^2)\log(4n/\tau)$ randomized queries to G with error prob. $\leq \tau$.
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Let $G = \left(n, 2, \{\bar{u}^i_j\}_{i \in [n], j \in \{1,2\}}\right)$ be an anonymous game.

For $1/\epsilon = O(n^{-1/4})$, Algorithm 3 can be used to find (with prob. $\geq 3/4$) an ϵ-NE of G, using $O(n^{3/2} \cdot \log^2 n)$ single-payoff queries in time $O(n^{3/2} \cdot \log^2 n)$.

- Setting $\delta = \zeta = 1/\sqrt[4]{n}$, $\tau = 1/(16 \log n)$, we find an $O(1/\sqrt[4]{n})$-NE using $O(\sqrt{n} \cdot \log^2 n)$ all-players queries w.p. $\geq 3/4$.
    - A family of algorithm parameterized by ϵ, i.e., the solutions to $\epsilon = \zeta + \delta + 1/(\zeta\sqrt{n}) \quad (\text{for } \epsilon \in [n^{-1/4}, 1))$.

An $\Omega(\log n)$ lower bound (for ϵ-WSNE)

Theorem 7.2

For any $\epsilon \in [0, 1)$, any randomized all-players query algorithm must make $\Omega(\log n)$ queries to find an ϵ-WSNE of $\mathcal{G}_n$ in the worst case.

The following work

- Yu Cheng, Ilias Diakonikolas, Alistair Stewart: Playing Anonymous Games using Simple Strategies. *arXiv* (25 Aug. 2016).

Donald E. Knuth Prize 2016

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Noam Nisan

School of Computer Science and Engineering,
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ACM Awards Knuth Prize to Pioneer of Algorithmic Game Theory

Hebrew University's Nisan Cited for Fundamental and Lasting Contributions to Theoretical Computer Science

New York, September 8, 2016 —The 2016 Donald E. Knuth Prize will be awarded to Noam Nisan of the Hebrew University of Jerusalem for fundamental and lasting contributions to theoretical computer science in areas including communication complexity, pseudorandom number generators, interactive proofs, and algorithmic game theory. The Knuth Prize is jointly bestowed by the ACM Special Interest Group on Algorithms and Computation Theory (SIGACT) and the IEEE Computer Society Technical Committee on the Mathematical Foundations of Computing (TCMF). It will be presented at the 57th Annual Symposium on Foundations of Computer Science (FOCS 2016) in New Brunswick, NJ, October 9–11.

Thank you.