Introduction to the Regularity Lemma

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Outline

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Introduction

Theorem 1.1 (Szemerédi's Theorom)

Let k be a positive integer and let $0 < \delta < 1$. Then there exists a positive integer $N = N(k, \delta)$, such that for every $A \subset \{1, 2, ..., N\}$, $|A| \geq \delta N$, A contains an arithmetic progression of length k.

- A branch of Ramsey theory (see also *Van der Waerden's theorem*).
- How about $N(3, 1/2)$?
  - $\{1, 2, 3, 4, 5, 6, 7, 8\}$
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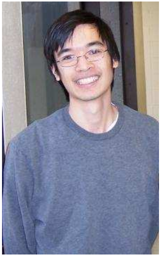
Related to a famous result...

Theorem

The primes contain arbitrarily long arithmetic progressions.
(Terence Tao and Ben J. Green, 2004)



Terence Tao (2006 Fields Medal)

Introduction (contd.)

- The best-known bounds for $N(k, \delta)$:
  - $C^{\log(1/\delta)^{k-1}} \leq N(k, \delta) \leq 2^{2^{\delta^{-2^{2^{k+9}}}}}$.
- The Regularity Lemma (Szemerédi 1978) was invented as an auxiliary lemma in the proof of Szemerédi's Theorom.
- Roughly speaking, every graph (large enough) can, in some sense, be approximated by (pseudo-)random graphs.
- Helpful in proving theorems for arbitrary graphs whenever the corresponding result is easy for random graphs.

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Introduction (contd.)

Endre Szemeréi

- About 15 years later, its power was noted and plenty results in graph theory and theoretical computer science have been worked out.

Outline

Density of bipartite graphs

Definition 2.1

Given a bipartite graph $G = (A, B, E)$, $E \subset A \times B$. The density of G is defined to be

$$d(A, B) = \frac{e(A, B)}{|A| \cdot |B|},$$

where $e(A, B)$ is the number of edges between A, B.

- A perfect matching of G has density $1/n$ if $|A| = |B| = n$.
- $d(A, B) = 1$ If G is complete bipartite.

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ϵ-regular pair

Definition 2.2

Let $\epsilon > 0$. Given a graph G and two disjoint vertex sets $A \subset V$, $B \subset V$, we say that the pair (A, B) is ϵ-regular if for every $X \subset A$ and $Y \subset B$ satisfying

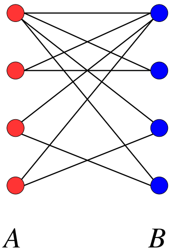
$$|X| \geq \epsilon|A| \text{ and } |Y| \geq \epsilon|B|,$$

we have

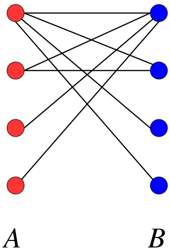
$$|d(X, Y) - d(A, B)| < \epsilon.$$

- If $G = (A, B, E)$ is a complete bipartite graph, then (A, B) is ϵ-regular for every $\epsilon > 0$.

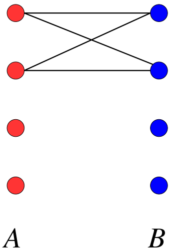
ϵ-regular pair (contd.)



1/2-regular



1/2-irregular



1/2-irregular

Regularity is preserved when moving to subsets

Fact 2.3

Assume that

- (A, B) *is a* ϵ*-regular and* $d(A, B) = d$*, and*
- $A' \subset A$ *and* $B' \subset B$ *satisfy* $|A'| \geq \gamma|A|$ *and* $|B'| \geq \gamma|B|$ *for some* $\gamma \geq \epsilon$*,*

then

- (A', B') *is a* $\max\{2\epsilon, \gamma^{-1}\epsilon\}$*-regular and*
- $d(A', B') \geq d - \epsilon$ *or* $d(A', B') \leq d + \epsilon$*.*

Proof of Fact 2.3

- Consider $A'' \subset A'$ and $B'' \subset B''$, s.t. $|A''| \geq \frac{\epsilon}{\gamma} \cdot \gamma|A'| \geq \epsilon|A|$ and $|B''| \geq \frac{\epsilon}{\gamma} \cdot \gamma|B'| \geq \epsilon|B|$.
    - $|d(A'', B'') - d(A, B)| < \epsilon$.
    - Hence $|d(A', B') - d(A'', B'')| < 2\epsilon$.
- Furthermore, since $|d(A', B') - d(A, B)| < \epsilon$,
    - $d - \epsilon < d(A', B') < d + \epsilon$.
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Most degrees into a large set are large

Fact 2.4

Let (A, B) be an ϵ-regular pair and $d(A, B) = d$. Then for any $Y \subset B$, $|Y| > \epsilon|B|$ we have

$$\#\{x \in A \mid deg(x, Y) \leq (d - \epsilon)|Y|\} \leq \epsilon|A|,$$

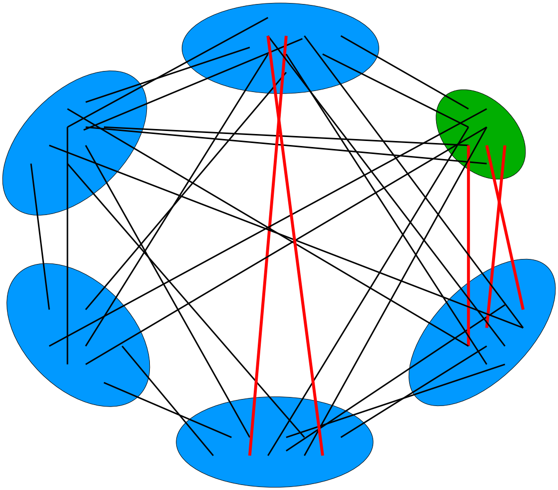
where $deg(x, Y)$ is the number of neighbors of x in Y.

Proof of Fact 2.4

- Let $\delta > \epsilon$ be a constant.
- Let $X = \{x \in A \mid \deg(x, Y) \leq (d - \epsilon)|Y|\}$.
- Assume $|X| = \delta|A| > \epsilon|A|$.
- Clearly $d(X, Y) \leq \frac{\delta|A| \cdot (d-\epsilon)|Y|}{\delta|A||Y|} \leq d - \epsilon$.
- But $d - \epsilon < d(X, Y)$ by the regularity of (A, B) and $|Y| > \epsilon|B|$.
- A contradiction occurs.

Outline

The famous Regularity Lemma



The famous Regularity Lemma (contd.)

Theorem 3.1 (Szemerédi's Regularity Lemma, 1978)

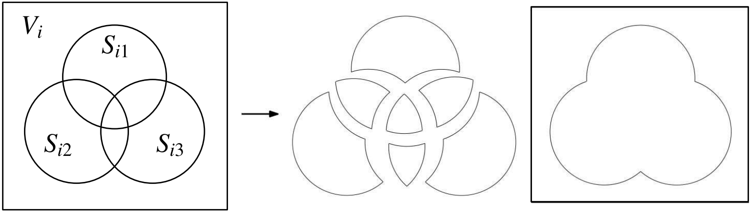
For every $\epsilon > 0$ and positive integer t, there exists two integers $M(\epsilon, t)$ and $N(\epsilon, t)$ such that

- *For every graph $G(V, E)$ with at least $N(\epsilon, t)$ vertices, there is a partition $(V_0, V_1, V_2, \ldots, V_k)$ of V with:*
  - $t \leq k \leq M(\epsilon, t)$,
  - $|V_0| \leq \epsilon n$, *and*
  - $|V_1| = |V_2| = \ldots = |V_k|$

  such that at least $(1 - \epsilon)\binom{k}{2}$ of pairs (V_i, V_j) are ϵ-regular.

One of the proofs...

- A combinatorial proof:



- k sets $\Rightarrow$ refine to $k \cdot 2^{k-1}$ sets $\rightarrow$ refine to $(k2^{k-1}) \cdot 2^{k2^{k-1}-1}$ $\Rightarrow \ldots$.
- A tower of 2's of height $O(1/\epsilon^5)$ (since $O(1/\epsilon^5)$ refinements required).
  - e.g., $2^{2^{2^{2^2}}}$: a tower of 2's of height 5.

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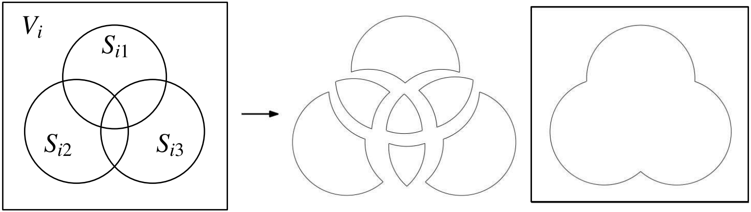

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Lower bound of $M(\epsilon, t)$ (k has to be in the worst case)

- The tower dependence on $1/\epsilon$ is necessary (by Timothy Gowers [4]).
- Constructive proof by Alon *et al.* [2]
  - $M(n) = O(n^{2.2376})$ time (matrix multiplication).
- "Deciding if a given partition of an input graph satisfies the property guaranteed by the regularity lemma" is co-**NP**-complete [2].

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Triangle Removal Lemma

Lemma 4.1 (Triangle Removal Lemma)

For all $0 < \delta < 1$, there exists $\epsilon = \epsilon(\delta)$, such that for every n-vertex graph G, at least one of the following is true:

1. *G can be made triangle-free by removing $< \delta n^2$ edges.*
2. *G has $\geq \epsilon n^3$ triangles.*

- We show this lemma by making use of the Regularity Lemma.

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Proof of the Triangle Removal Lemma

The regularity Lemma

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  such that at least $(1-\epsilon)\binom{k}{2}$ of pairs (V_i, V_j) are ϵ-regular.

- Let $\epsilon = \frac{\delta}{10}$ and $t = \frac{10}{\delta}$.
- Star with an arbitrary graph G $(n \geq N(\epsilon, t))$.
- Find a $\frac{\delta}{10}$-regular partition into $k = k(\frac{\delta}{10}, \frac{10}{\delta})$ blocks.

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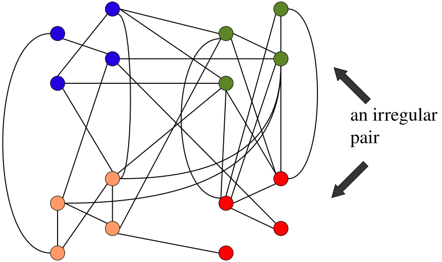
Proof of the Triangle Removal Lemma (contd.)

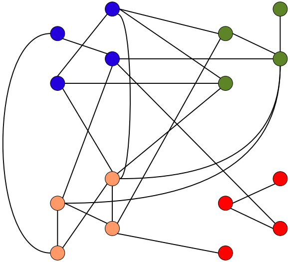
- Using the partition we just obtained, we define a reduced graph G' as follows:



Proof of the Triangle Removal Lemma (contd.)

I: Remove all edges between non-regular pairs (at most $\frac{\delta}{10}n^2$ edges).

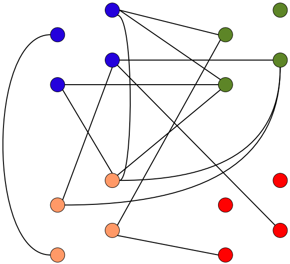
- $\leq \frac{\delta}{10}\binom{k}{2}$ irregular pairs, and at most $(\frac{n}{k})^2$ edges between each pair.



Proof of the Triangle Removal Lemma (contd.)

II: Remove all edges inside blocks (at most $\frac{\delta}{10}n^2$ edges).

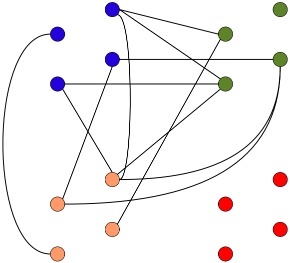
- k blocks, and each contains at most $\binom{n/k}{2}$ edges,
- $t \leq k$
- $\leq \frac{n^2}{k} \leq \frac{\delta}{10}n^2$ edges are removed.



Proof of the Triangle Removal Lemma (contd.)

III: Remove all edges between pairs of density $< \frac{\delta}{2}$ (at most $\frac{\delta}{2}n^2$ edges).

- $\leq \frac{\delta}{2}(\frac{n}{k})^2$ edges between a pair of density $< \frac{\delta}{2}$, and at most $\binom{k}{2}$ such pairs.



Proof of the Triangle Removal Lemma (contd.)

- Totally at most $(\delta/10 + \delta/10 + \delta/2)n^2 < \delta n^2$ edges are removed.
- Thus if G' contains no triangle, the first condition of the lemma is satisfied.
- Hence we suppose that G' contains a triangle and continue to see the second condition of the lemma.

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Proof of the Triangle Removal Lemma (contd.)

- By some technical reasons, we may assume $V_0 = \emptyset$ and let $m = n/k$ be the size of the blocks $(V_1, V_2, \ldots, V_k)$.
- A triangle in G' must go between three different blocks, say A, B, and C.
- If there is an edge between A and $B \Rightarrow$ there must be many edges (by Step III).

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Proof of the Triangle Removal Lemma (contd.)

- Since "most degrees into a large set are large"
  - $\leq m/4$ vertices in A have $\leq \frac{\delta}{4}m$ neighbors in B
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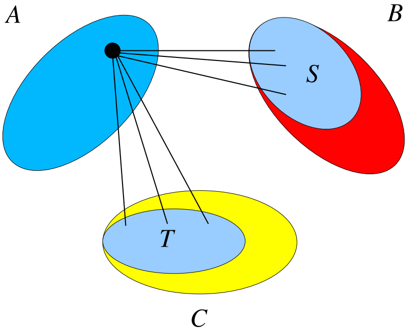
- Hence $\geq m/2$ vertices in A have both $\geq \frac{\delta}{4}m$ neighbors in B and $\geq \frac{\delta}{4}m$ neighbors in C.



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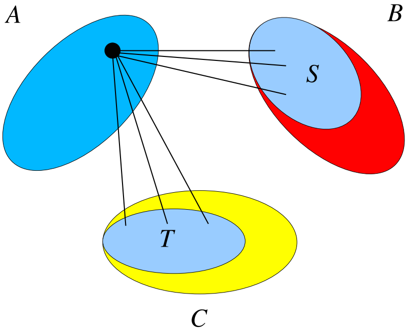
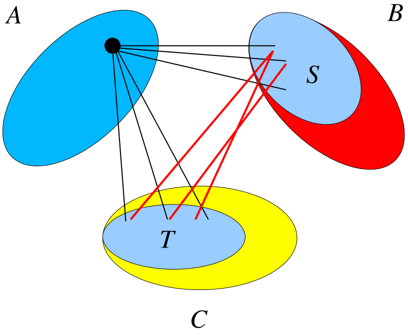


Proof of the Triangle Removal Lemma (contd.)

- Consider a such vertex from A.
- How many edges go between S and T?
  - $S \geq \frac{\delta}{4}m$ and $T \geq \frac{\delta}{4}m$
  - $d(B, C) \geq \frac{\delta}{2}$ and (B, C) is $\frac{\delta}{10}$-regular
  - hence $e(B, C) \geq (\frac{\delta}{2} - \frac{\delta}{10})|S||T| \geq \frac{\delta^3}{64}m^2$
- Total # triangles $\geq \frac{\delta^3}{64}m^2 \cdot \frac{m}{2} = \frac{\delta^3}{128k^3}n^3$.



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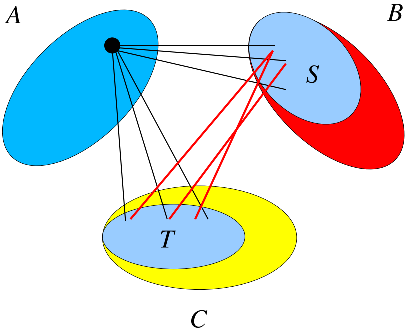

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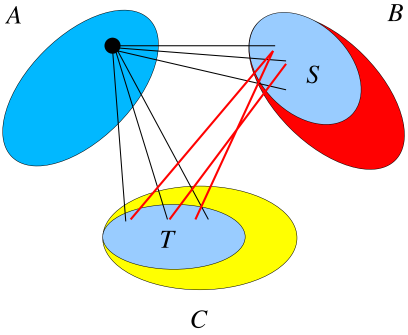
  - hence $e(B, C) \geq (\frac{\delta}{2} - \frac{\delta}{10})|S||T| \geq \frac{\delta^3}{64}m^2$
- Total # triangles $\geq \frac{\delta^3}{64}m^2 \cdot \frac{m}{2} = \frac{\delta^3}{128k^3}n^3$.



Is the Triangle Removal Lemma important? YES!

The Triangle Removal Lemma

For all $0 < \delta < 1$, there exists $\epsilon = \epsilon(\delta)$, such that for every n-vertex graph G, at least one of the following is true:

1. G can be made triangle-free by removing $< \delta n^2$ edges.
2. G has $\geq \epsilon n^3$ triangles.

- The graph property "triangle-free" is "testable".
- Yet the complexity has dependence of towers of δ.
  - e.g., $\frac{128k^3}{\delta^3}$, k is tower of 2's of height depending on $O(1/\delta)$.

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Conclusion and remarks

- A LOT OF applications of the Regularity Lemma in the field of property testing.
  - Counting the number of forbidden subgraphs, testing monotone graph properties, dealing with partition-type problems, etc.
- Excellent surveys for the Regularity Lemma: [5, 6]; and nice lecture notes: [1] (by Luca Trevisan); also Luca Trevisan's Blog: "in theory" (http://lucatrevisan.wordpress.com).

Question

Is it possible to apply the Regularity Lemma to design fixed-parameter algorithms for graph problems?

Thank you!

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