Computing branching numbers

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Branching vector

Definition 1

Given a problem $\mathbb{P}$ with parameter k. If an algorithm solves $\mathbb{P}$ and calls itself recursively for subproblems with parameters

$$k - d_1, k - d_2, \ldots, k - d_i,$$

then $(d_1, d_2, \ldots, d_i)$ is called the *branching vector* of recursion of the algorithm.

- Actually, the branching vector $(d_1, d_2, \ldots, d_i)$ corresponds to the recurrence $T_k = T_{k-d_1} + T_{k-d_2} + \ldots + T_{k-d_i}$.

- $T_k = T_{k-d_1} + T_{k-d_2} + \ldots + T_{k-d_i}$.
- Let $T_k = \alpha^k$. We want to know how big α is.
- $\alpha^k = \alpha^{k-d_1} + \alpha^{k-d_2} + \ldots + \alpha^{k-d_i}$.
  - $1 = \alpha^{-d_1} + \alpha^{-d_2} + \ldots + \alpha^{-d_i}$.
- Let $d = \max\{d_1, d_2, \ldots, d_i\}$. We have

$$\alpha^d = \alpha^{d-d_1} + \alpha^{d-d_2} + \ldots + \alpha^{d-d_i}.$$

Characteristic polynomials

Definition 2

Given a branching vector $\mathbf{v} = (d_1, d_2 \ldots, d_i)$ of some recursion, the *characteristic polynomial* of $\mathbf{v}$ is

$$z^d - z^{d-d_1} - z^{d-d_2} - \ldots - z^{d-d_i},$$

where d is defined to be $\max\{d_1, d_2, \ldots, d_i\}$. Furthermore, we call α the *characteristic root* of the characteristic polynomial if $\alpha^d = \alpha^{d-d_1} + \alpha^{d-d_2} + \ldots + \alpha^{d-d_i}$.

Reflect characteristic polynomials

Definition 3

Given a branching vector $\mathbf{v} = (d_1, d_2, \ldots, d_i)$ of some recursion, the *reflected characteristic polynomial* of $\mathbf{v}$ is $1 - z^{d_1} - z^{d_2} - \ldots - z^{d_i}$.

- For example, recall that $\alpha^k = \alpha^{k-d_1} + \alpha^{k-d_2} + \ldots + \alpha^{k-d_i}$.
  - $1 = \alpha^{-d_1} + \alpha^{-d_2} + \ldots + \alpha^{-d_i}$.
  - Let $\beta = 1/\alpha$, we have $1 = \beta^{d_1} + \beta^{d_2} + \ldots + \beta^{d_i}$.
- Advantages of using reflected characteristic polynomials:
  - It's easy to memorize.
  - The root of a reflected characteristic polynomial is always in $(0, 1]$.

▷ Note: Do not forget to take the inverse of the root.

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Calculation the branching number by a program

- Let $f(z) = 1 - z^{d_1} - z^{d_2} - \ldots - z^{d_i}$ where $\mathbf{v} = (d_1, d_2, \ldots, d_i)$ is a branching vector.
- $f(z)$ is a monotonically decreasing function for all $z \geq 0$.
- **Fact:** For $0 < a < b < 1$, if $f(a) > 0$ and $f(b) < 0$, then there exists a number $a < c < b$ such that $f(c) = 0$.
  - ⋆ $f(0.1) > 0$, $f(0.2) > 0$, $f(0.3) > 0$, $f(0.4) < 0$.
  - ⋆ $f(0.31) > 0$, $f(0.32) > 0$, $f(0.33) > 0$, $f(0.34) > 0$, $f(0.35) > 0$, $f(0.36) < 0$.
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  - Then the root of the corresponding reflected characteristic polynomial is $0.35\ldots$. Thus the branching number is less than $1/0.35 = 2.85714286\ldots$.

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- For example, assume that we have $T(k) = 2T(k-1) + T(k-3) + T(k-5)$.
- The branching vector: $(1, 1, 3, 5)$.
- The characteristic polynomial: $z^5 - 2z^4 - z^2 - 1$.
- The reflected characteristic polynomial: $1 - 2z - z^3 - z^5$.
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An special branching number

- What is the branching number of $(1, 2)$?

- $f(z) = 1 - z - z^2$.
- Solve the equation $z^2 + z - 1 = 0$.
  - $z^* = \frac{-1 \pm \sqrt{5}}{2} \approx 0.61803399$ (take the positive one).
- An interesting fact: $1/z^* = 1 + z^*$.
  - $\frac{1}{z^*} = \frac{2}{\sqrt{5}-1} = \frac{2(\sqrt{5}+1)}{(\sqrt{5}+1)(\sqrt{5}-1)} = \frac{\sqrt{5}+1}{2} = 1 + z^*$.

Golden ration and spirals

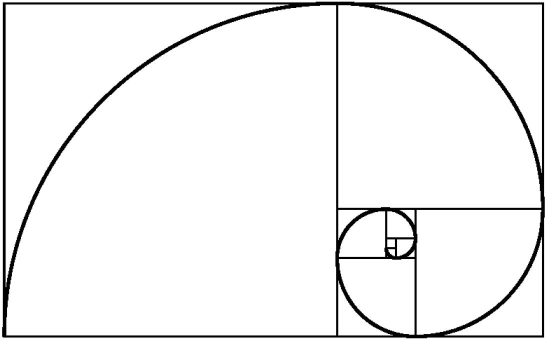

Fig.: The approximate golden spiral.

Demostration

- http://www.cs.ccu.edu.tw/~lincc/Program/br.c
- http://www.cs.ccu.edu.tw/~lincc/Program/br.exe

Thank you!