Equitable coloring extends Chernoff-Hoeffding bounds

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- In 1952, Herman Chernoff introduced a technique which gives sharp upper bounds on the *tails* of the distribution of the sum of mutually independent binary (Bernoulli) random variables.
- Wassily Hoeffding extended Chernoff's technique to deal with bounded independent random variables.
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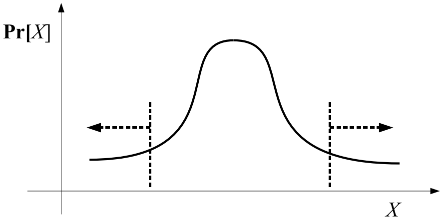

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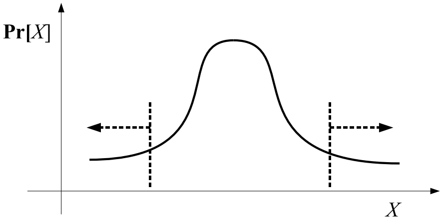

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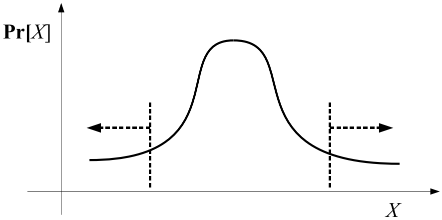

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- CH bounds are extremely useful in design and analysis of randomized algorithms, in proofs by the probabilistic method, analysis in computational complexity, etc.
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Chernoff-Hoeffding bounds

- Let $\mathcal{X} = \{X_1, X_2, \ldots, X_n\}$ denote a set of *mutually independent Bernoulli random variables* with $S = \sum_{i=1}^{n} X_i$ and $\mu = \mathbf{E}[S]$.
  - Assume that, for all i, $\mathbf{Pr}[X_i = 1] = p$ for some $p > 0$.

- We are interested in upper bounds on $\mathbf{Pr}[S \geq (1+\delta)\mu]$ and $\mathbf{Pr}[S \leq (1-\delta)\mu]$.

- Chernoff bounds lead to

$$\begin{aligned}
\mathbf{Pr}[S \geq (1+\delta)\mu] &\leq \left(\frac{e^{\delta}}{(1+\delta)^{(1+\delta)}}\right)^{\mu}, \quad \stackrel{\text{def}}{=} F^{+}(\mu,\delta) \\
\mathbf{Pr}[S \leq (1-\delta)\mu] &\leq e^{-\mu\delta^2/2}. \quad \stackrel{\text{def}}{=} F^{-}(\mu,\delta)
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When $\delta \leq 1$, we can derive $F^{+}(\mu,\delta) \leq e^{-\mu\delta^2/3}$.

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A simple application (a generous teacher and diligent students)

- There are n students, who work very hard all the time just like us. Their teacher, who is very generous, would like to reward them.

- In front of them, there is a sealed box which has 3 golden balls and 1 black ball inside.
- Each time one can pick a ball from the box and then put it back into the box (we assume that the students are honest).
- The teacher said he will treat the students a bountiful feast if more than $n/2$ students get golden balls.

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A simple application (a generous teacher and diligent students) (contd.)

- For $i = 1, \ldots, n$, $X_i = 1$: the ith student gets a golden ball; $X_i = 0$: the ith student gets a black ball.
- $\mathbf{Pr}[X_i = 1] = 3/4$ and $\mathbf{Pr}[X_i = 0] = 1/4$.
- Let $S = \sum_{i=1}^{n} X_i$. The event that the students have bad luck is $S \leq n/2$, and we have $\mu = \mathbf{E}[S] = 3n/4$.
- $\mathbf{Pr}[S \leq n/2] = \mathbf{Pr}[S \leq (1 - 1/3)\mu] \leq e^{-\mu(1/3)^2/2} = e^{-n/24}$.
- The probability is less than 0.66 if $n = 10$, less than 0.125 if $n = 50$, and less than 0.005 if $n = 130$.

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Hoeffding's extension

- Consider the case that X_i's are mutually independent "bounded" random variables (i.e., $a_i \leq X_i \leq b_i$, for some positive real a_i and b_i).
- Hoeffding's extension of Chernoff's technique:

$$\mathbf{Pr}[|S - \mu| \geq \delta\mu] \leq 2e^{-2\mu^2\delta^2 / \sum_{i=1}^{n}(b_i - a_i)^2}.$$

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The crucial step and limitation of CH bounds

- A crucial step for deriving CH bounds is to calculate $\mathbf{E}[e^{tS}]$ for any positve real t (the moment generating function).

$$\mathbf{E}[e^{tS}] = \mathbf{E}[e^{t\sum_{i=1}^{n} X_i}] = \mathbf{E}\left[\prod_{i=1}^{n} e^{tX_i}\right] = \prod_{i=1}^{n} \mathbf{E}[e^{tX_i}].$$

- The last of the above equalities depends on the X_i's being mutually independent.
- This is the limitation for CH bounds.
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Some basic definitions

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- A is said to be mutually independent of a set of events $B_1, B_2, \ldots, B_n$ if for any $I \subseteq \{1, 2, \ldots, n\}$, $\mathbf{Pr}[A \mid \bigcap_{j \in I} B_j] = \mathbf{Pr}[A]$.

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Dependency graphs

- $\mathcal{X} = \{X_1, X_2, \ldots, X_n\}$: a set of random variables.
- A dependency graph $G = (V, E)$ for $\mathcal{X}$ has a vertex set $[n] = \{1, 2, \ldots, n\}$ and for each i, X_i is mutually independent of the events $\{X_j \mid (i, j) \notin E\}$.
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- Let G be a dependency graph of $\mathcal{X}$.
- Assume that $X_1, X_2, \ldots, X_k$ correspond to an independent set of G.

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  - X_1: the first flip is head;
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- The events can be shown to be pairwise independent for each two of them.

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- ANY graph with three vertices and at least two edges is a dependency graph of $\mathcal{X}$.

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The main theorem

Theorem 1

For identically distributed Bernoulli random variables X_i with d-bounded dependence, for any $0 < \delta \leq 1$, we have the upper tail probability bound

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and the lower tail probability bound

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- Given a k-regular n-vertex graph G. The following steps compute a large independent set in G.

Step 1: Delete each vertex from G independently with probability $1 - 1/k$.

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An example: a randomized algorithm for Maximum Independent Set in a regular graph (contd.)

- The size of the independent set computed by the algorithm: $\geq A - B$.
- Hence the expected size of the solution produced by the algorithm is $\geq n/2k$.
  - A randomized $O(1)$-factor approximation algorithm for Maximum Independent Set.

An example: a randomized algorithm for Maximum Independent Set in a regular graph (contd.)

- Actually we can show that $A - B$ is very close to $n/2k$ with high probability.
- It is clear that A_i's are mutually independent, so CH bounds can be applied.
- However, B_i's are NOT mutually independent.
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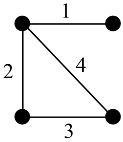
An example: a randomized algorithm for Maximum Independent Set in a regular graph (contd.)

- The line graph (i.e., edge graph) $L(G)$ of G is a dependency graph of the B_i's.
  - $L(G)$: every vertex of $L(G)$ represents an edge of G, and two vertices of $L(G)$ are adjacent iff their corresponding edge share a common endpoint in G.



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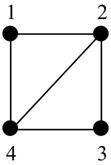

$L(G)$

An example: a randomized algorithm for Maximum Independent Set in a regular graph (contd.)

- G is k-regular $\longrightarrow$ $L(G)$ is $2(k-1)$-regular $\longrightarrow$ B_i's exhibit $2(k-1)$-bounded dependence.
- $\mathbf{E}[B]/(2k-1) = \Omega(n)$.
  - $\Omega(\log^{1+\rho} kn/2)$ for any $\rho > 0$.
- Thus the main theorem of this paper can be applied, and then we know the algorithm indeed produces a large independent set with high probability.

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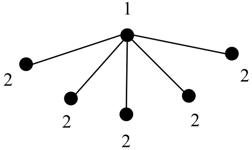
Outline

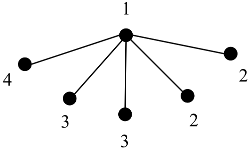
t-equitable coloring

Definition 2

A coloring of a graph is equitable if the sizes of any pair of color classes are within one of each other.

- t-equitable coloring: an equitable coloring using t colors.





A deep result by Hajnal and Szemerédi

Hajnal-Szemerédi (1970)

A graph G with maximum degree Δ has a $(\Delta + 1)$-equitable coloring.

Lemma 3

Suppose that X_i's are identical Bernoulli random variables with dependency graph G, and suppose G has a t-equitable coloring. Then for any $0 < \delta \leq 1$, we have

$$\begin{aligned}\mathbf{Pr}[S \geq (1+\delta)\mu] &\leq \frac{4t}{e} F^{+}(\mu,\delta)^{1/t},\\ \mathbf{Pr}[S \leq (1-\delta)\mu] &\leq \frac{4t}{e} F^{-}(\mu,\delta)^{1/t}.\end{aligned}$$

Theorem 4

Suppose the X_i's are identical Bernoulli random variables exhibiting d-bounded dependence. Then, for any $0 < \delta \leq 1$, we have

$$\begin{aligned}\mathbf{Pr}[S \geq (1+\delta)\mu] &\leq \frac{4(d+1)}{e} F^{+}(\mu,\delta)^{\frac{1}{d+1}},\\ \mathbf{Pr}[S \leq (1-\delta)\mu] &\leq \frac{4(d+1)}{e} F^{-}(\mu,\delta)^{\frac{1}{d+1}}.\end{aligned}$$

Proof of Lemma 3

- For convenience, assume that $\mathbf{E}[X_i] = \mu'$ for each i, and let $[t]$ denote $\{1, 2, \ldots, t\}$.
- Let $C_1, C_2, \ldots, C_t$ be the t color classes in a t-equitable-coloring of G.
- For each $i \in [t]$, let $\mu_i = \mathbf{E}[\sum_{j \in C_i} X_j]$ (i.e., $\mu' |C_i|$).

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Proof of Lemma 3 (contd.)

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\begin{aligned}
S \geq (1+\delta)\mu \quad &\equiv \quad S \geq (1+\delta)\mu' n \\
&\equiv \quad S \geq (1+\delta)\mu' \sum_{i\in[t]} |C_i| \\
&\equiv \quad S \geq \sum_{i\in[t]} (1+\delta)\mu' |C_i| \\
&\equiv \quad \sum_{i\in[t]} \sum_{j\in C_j} X_j \geq \sum_{i\in[t]} (1+\delta)\mu_i .
\end{aligned}
$$

- The first equivalence: $\mu = \mathbf{E}[\sum_{i\in[n]} X_i] = \sum_{i\in[n]} \mathbf{E}[X_i] = n\mu'$.
- The second equivalence: C_i's form a partition of $[n]$.
- The last equivalence: expressing S as the sum of the X_i's grouped into color classes.

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Proof of Lemma 3 (contd.)

- $\sum_{i\in[t]}\sum_{j\in C_i} X_j \geq \sum_{i\in[t]}(1+\delta)\mu_i \Rightarrow \exists i \in [t]\colon \sum_{j\in C_i} X_j \geq (1+\delta)\mu_i$.
- Hence

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- The last probability above is actually at most

$$\begin{aligned}&\sum_{i\in[t]} \mathbf{Pr}\left[\sum_{j\in C_i} X_j \geq (1+\delta)\mu_i\right] \quad \text{(union bound)} \\ \leq\ &\sum_{i\in[t]} F^+(\mu_i,\delta) \quad \text{(Chernoff bound)}.\end{aligned}$$

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- The last inequality: $(1+\delta)^{1+\delta}/e^\delta$ is a monotonically increasing function of δ and its maximum occurs when $\delta = 1$.
- Thus the upper tail probability is proved.

$$\mathbf{Pr}[S \geq (1+\delta)\mu] \leq \frac{4t}{e} F^+(\mu,\delta)^{1/t}.$$

  - The proof of the lower tail probability is identical.

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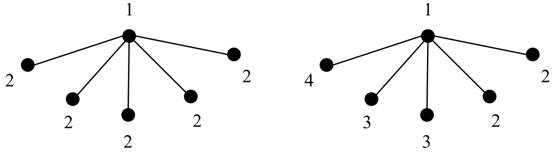
Outline

Equitable chromatic number $\chi_{eq}(G)$

- $\chi(G)$: the chromatic number of G.
- $\chi_{eq}(G)$: the fewest colors required to equitably color the graph G.
- E.g., $\chi(G) = 2$ and $\chi_{eq}(G) = \lceil (n-1)/2 \rceil + 1$ when G is an n-vertex star graph.



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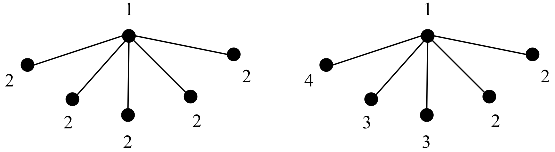

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Bollobás-Guy (1983)

A tree T with n vertices is equitably 3-colorable if $n \geq 3\Delta(T) - 8$ or if $n = 3\Delta(T) - 10$.

- The theorem implies that if $\Delta(T) \leq n/3$, then T can be equitably 3-colored. Thus we have

Theorem 5

Suppose that X_i's are identical Bernoulli random variables such that the corresponding dependency graph is a tree with maximum degree at most $n/3$. Then we have the following bounds

$$\begin{aligned} \mathbf{Pr}[S \geq (1+\delta)\mu] &\leq \frac{12}{e} F^{+}(\mu,\delta)^{1/3}, \\ \mathbf{Pr}[S \leq (1-\delta)\mu] &\leq \frac{12}{e} F^{-}(\mu,\delta)^{1/3}. \end{aligned}$$

Pemmaraju (2001); technical report

A connected outerplanar graph with n vertices and vertex degree at most $n/6$ has a 6-equitable coloring.

Theorem 6

Suppose that X_i's are identical Bernoulli random variables whose dependency graph is outerplanar with maximum degree at most $n/6$. Then we have the following bounds

$$\begin{aligned}\mathbf{Pr}[S \geq (1+\delta)\mu] &\leq \frac{24}{e} F^{+}(\mu,\delta)^{1/6}, \\ \mathbf{Pr}[S \leq (1-\delta)\mu] &\leq \frac{24}{e} F^{-}(\mu,\delta)^{1/6}.\end{aligned}$$

Some further remarks

- Are the bounds on the vertex-degree required to obtain sharp bounds?
- a (c, α)-coloring: a vertex coloring such that
  - $\leq c$ vertices are not colored.
  - for any pair of color classes C and C', $|C| \leq \alpha|C'|$.
- It is possible to extend Bollobás-Guy Theorem to have the following results.

Theorem 7

Every tree has a $(1, 5)$-coloring with two colors.
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Thank you!