Complement reducible graphs

D. G. Corneil, H. Lerchs, and L. S. Burlingham

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Goals of this talk

This talk includes the following goals:

- Briefly introduce coplement reducible graphs (cographs).
- Main results of the paper by Corneil et al.
  - Equivalent definitions for cographs.
  - Structure properties and algorithmic properties of cographs.
- Proofs done by myself.

It'd be better for you to listen to the speaker instead of looking at the slides.

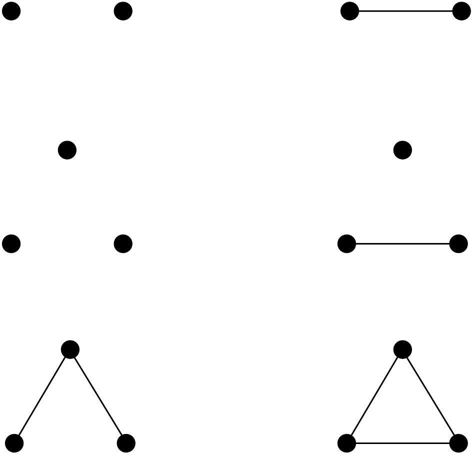
Outline

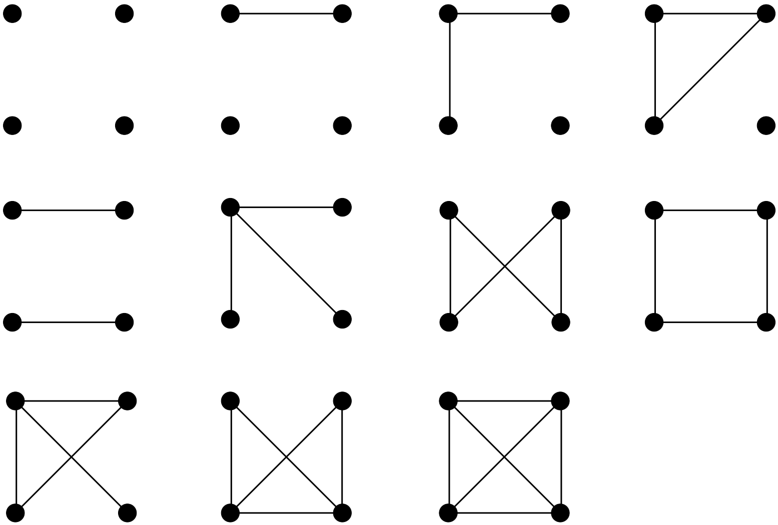
Outline

Introduction

A complement reducible graph (cograph) is defined recursively as follows:

- A graph with only one vertex is a cograph.
- If $G_1, G_2, \ldots, G_k$ are cographs, then so is their union $G_1 \cup G_2 \cup \ldots \cup G_k$.
- If G is a cograph, then so is its complement $\bar{G}$.



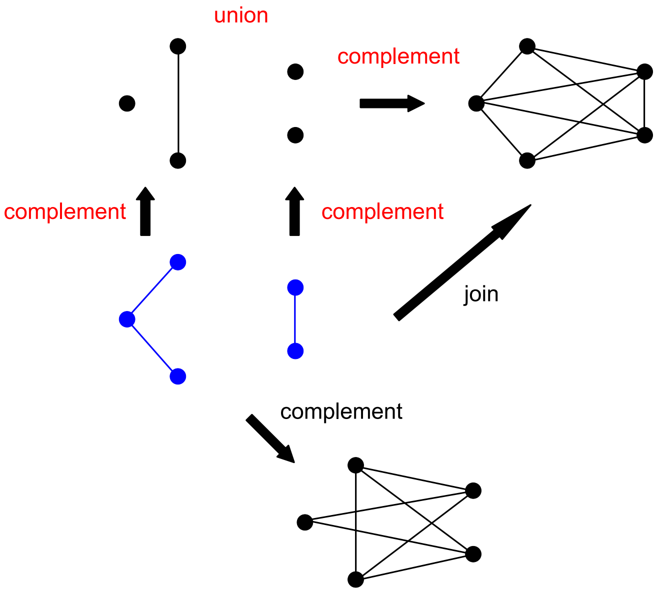


Introduction (contd.)

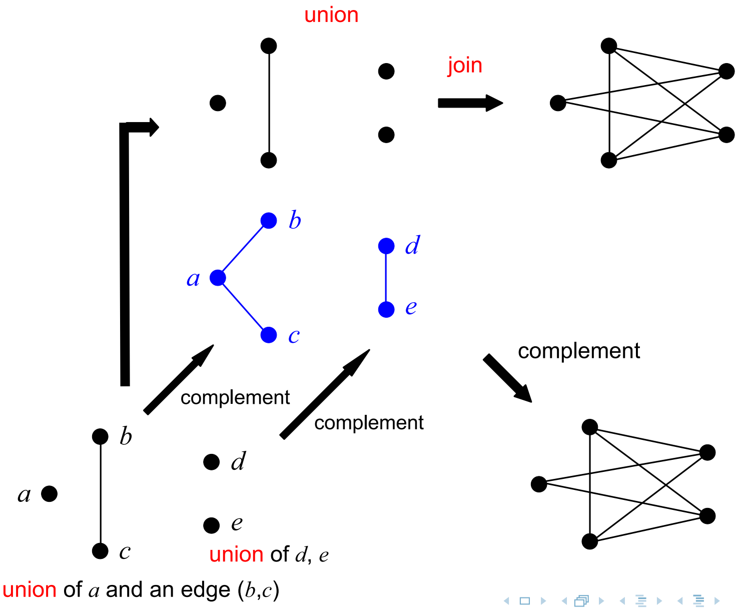
Another equivalent definition for cographs.

- A graph with only one vertex is a cograph.
- If $G_1, G_2, \ldots, G_k$ are cographs, then so is their union $G_1 \cup G_2 \cup \ldots \cup G_k$.
- If $G_1, G_2, \ldots, G_k$ are cographs, then so is their join.

Introduction (contd.)



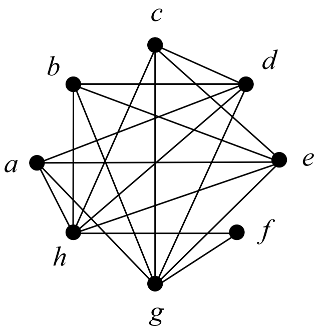
Introduction (contd.)

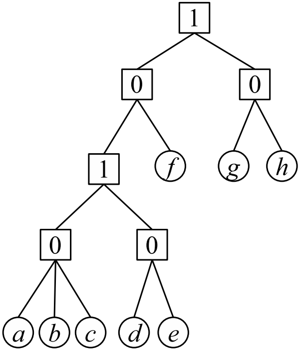


Introduction (contd.)

A tree representing union and join operations of construction of cographs is called a cotree.

Introduction (contd.)





Outline

Terminology (graphs and twins)

- Graph $G = (V, E)$;
  - V: the set of vertices of G;
  - E: the set of edges of G.
- $\forall x \in V$, $N(x) = \{y \in V \mid (x, y) \in E\}$.
- x, y are twins if $N(x) \setminus \{x, y\} = N(y) \setminus \{x, y\}$;
  - true twins if $(x, y) \in E$; false twins if $(x, y) \notin E$.

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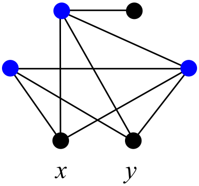
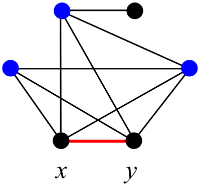


false twins x, y



true twins x, y

Terminology (kernels and cliques)

- kernel: a maximal independent set of G.
- clique: a maximal complete set of G.
  - $S \subseteq V$ is a kernel in $G \Leftrightarrow S$ is a clique in $\bar{G}$
- $\mathcal{C}_G$: the set of cliques of G.
  - $\mathcal{C}_G(x)$: the set of cliques of G containing x;
  - $\mathcal{C}_G(\bar{x})$: the set of cliques of G NOT containing x.
- $\mathcal{K}_G$: the set of kernels of G.
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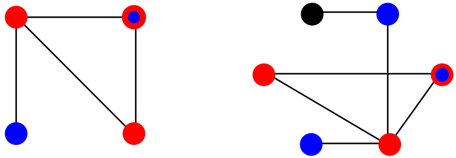
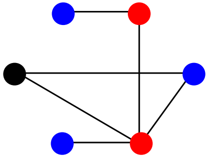
Terminology (CK-property)

A graph G has the clique-kernel intersection property (CK-property) iff $\forall C \in \mathcal{C}_G$ and $\forall K \in \mathcal{K}_G$, $|C \cap K| = 1$.



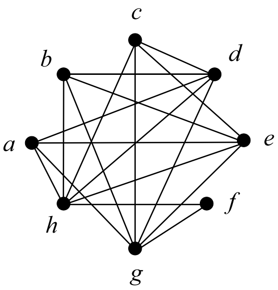
Terminology (CK-property contd.)

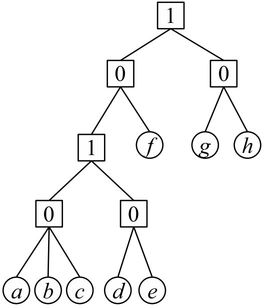


A ha!

Outline

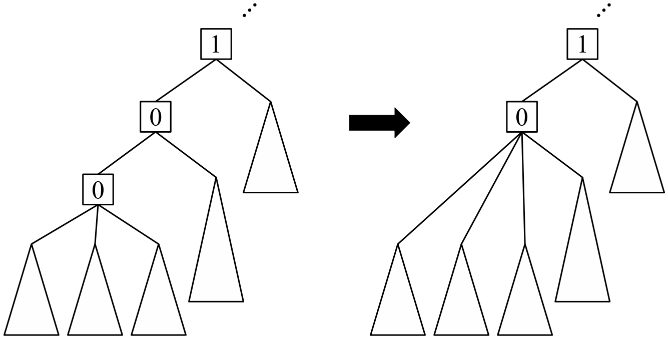
Go back to cotrees



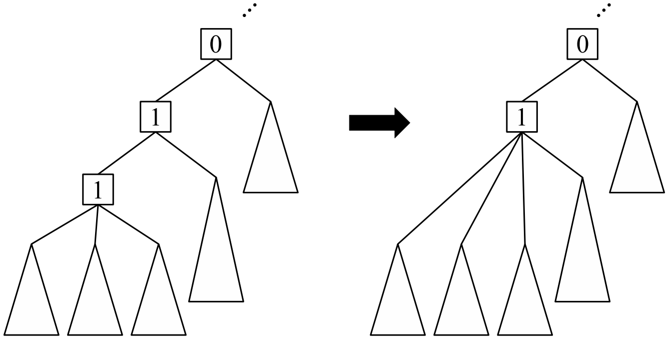


- A standard cotree of a cograph:
  - The internal nodes on each root-to-leaf path on the treee alternate between 0 and 1.
  - Each internal node has at least two children (except the trivial cograph).
- Uniqueness.

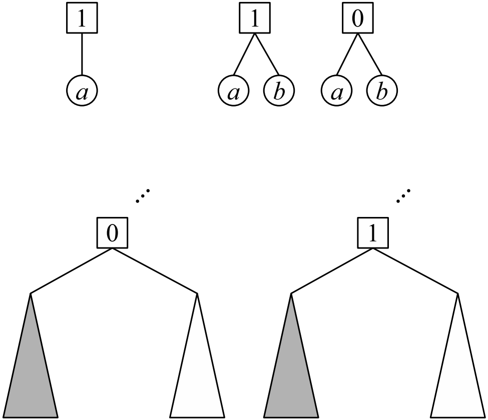
Root-to-leaf paths with alternating 0's and 1's



Root-to-leaf paths with alternating 0's and 1's

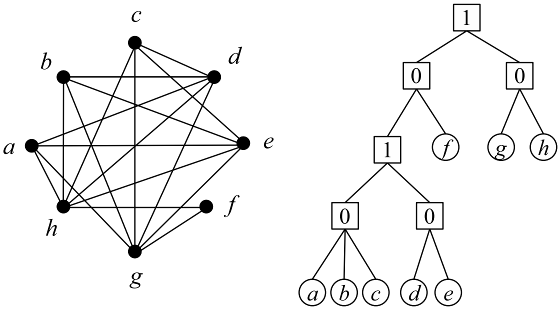


Uniqueness



Given a cograph G and its standard cotree T.

- $(x, y) \in E$ iff their lowest common ancestor in T is a 1-node (join).



Being a cograph is hereditary

Theorem

Every induced subgraph of a cograph is a cograph.

Proof.

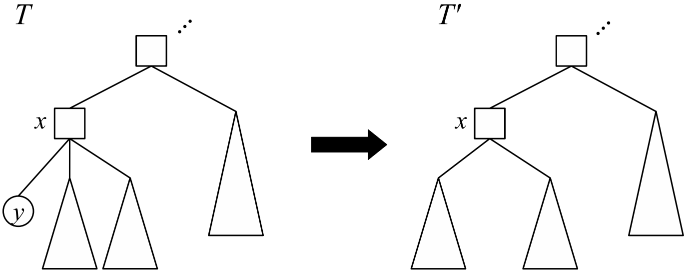
- In the following we assume that $n \geq 3$.
  - Obviously true for $n < 3$.
- **Fact:** Any induced subgraph of a graph can be obtained by removing vertices one by one.

Being a cograph is hereditary (contd.)

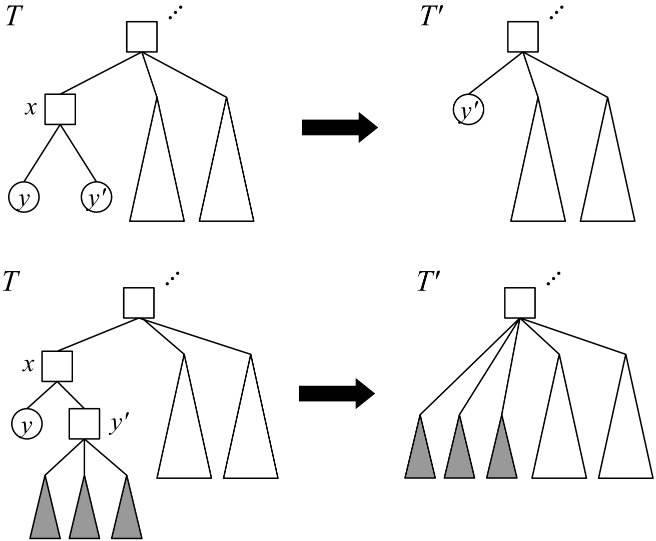
Proof (contd.)

- Let $G = (V, E)$ be a cograph and T be the associated cotree.
- The induced subgraph G' on $V \setminus \{y\}$ is a cograph $\Leftrightarrow$ a cotree T' can be associated with it.

Being a cograph is hereditary (contd.)



Being a cograph is hereditary (contd.)



Main theorem

Theorem

Given a graph G, the following statements are equivalent:

(1) *G is a cograph.*

(2) *Any nontrivial induced subgraph of G has at least one pair of twins.*

(3) *Any induced subgraph of G has the CK-property.*

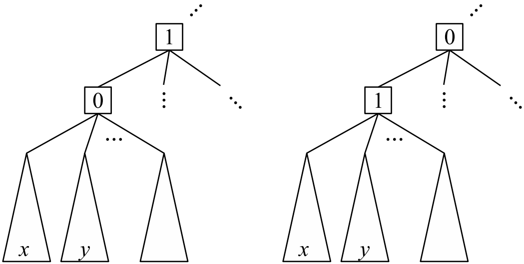
(4) *G does not contain P_4 as an induced subgraph.*

(5) *The complement of any nontrivial connected induced subgraph of G is disconnected.*

Proof of the main theorem: (1) $\Rightarrow$ (2)

G is a cograph $\Rightarrow$ any nontrivial induced subgraph of G has at least one pair of twins.

- It's sufficient to show that any cograph G (for $|V| \geq 2$) has at least one pair of twins.
- Obviously true by examining the leaves of the associated cotree T.



Proof of the main theorem: (2) $\Rightarrow$ (3)

Any nontrivial induced subgraph of G has at least one pair of twins
$\Rightarrow$ any induced subgraph of G has the CK-property.

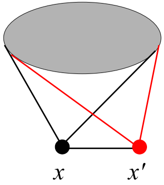
- Let p be the order of the induced subgraph in G.
- For $p = 1$: obviously true. (induction basis)
- Assume that all induced subgraphs of order p in G have the CK-property.
  - **Goal:** Show that any induced subgraph H of order $p + 1$ satisfies the CK-property.

Let x, x' be a pair of twins in H (existence guaranteed by (2)).
Let $H' = H - \{x'\}$ be the induced subgraph of H with x' removed.

(i): x, x' are true twins.

- $\mathcal{C}_H(\bar{x}) = \mathcal{C}_{H'}(\bar{x})$, $\mathcal{C}_H(x) = \mathcal{C}_{H'}(x) + \{x'\}$.
- $\mathcal{K}_H(\bar{x'}) = \mathcal{K}_{H'}$, $\mathcal{K}_H(x') = \mathcal{K}_{H'}(x) - \{x\} + \{x'\}$.

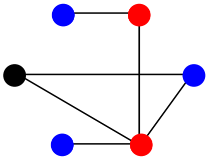
$\triangleright$ $\forall C \in \mathcal{C}_H$ and $\forall K \in \mathcal{K}_H$, we have $|C \cap K| = 1$.



(ii): x, x' are false twins.

- Identical to Case (i).

Any induced subgraph of G has the CK-property $\Rightarrow$ G does not contain P_4 as an induced subgraph.



A ha!

Proof of the main theorem: (4) $\Rightarrow$ (5)

G does not contain P_4 as an induced subgraph $\Rightarrow$ the complement of any nontrivial connected induced subgraph of G is disconnected.

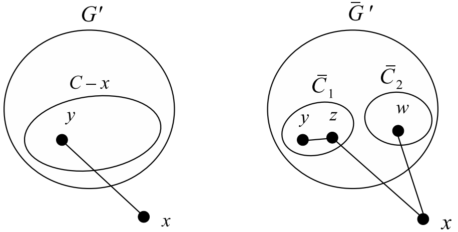
- Trivially true for $n \leq 3$.
- Assume that the argument holds when $n < p \nless 4$.
- Let $|V(G)| = p$ and $G' = G - \{x\}$.

  (i): x is not adjacent to any vertex in G'.
  - All nontrivial connected induced subgraphs of G are in G'.
  - By the inductive hypothesis the argument is true.

Proof of the main theorem: (4) $\Rightarrow$ (5)

(ii): x is not isolated in G (proof by contradiction).

- C: a connected component of G such that $\bar{C}$ is still connected;
  - x must be in C.
- y: a neighbor of x in G';
- $\bar{C} - x = \bar{C}_1 \cup \bar{C}_2$;
  - $\bar{C} - x$ is disconnected (by the inductive hypothesis).
  - WLOG assume that $y \in V(\bar{C}_1)$.
- $z \in V(\bar{C}_1) \setminus \{y\}$, $w \in V(\bar{C}_2)$.



Proof of the main theorem: (5) $\Rightarrow$ (1)

The complement of any nontrivial connected induced subgraph of G is disconnected $\Rightarrow$ G is a cograph.

G has the CCD property: the complement of any nontrivial connected component of G is disconnected.

- **Claim:** If G has the CCD property then so does $\bar{G}$.
  - Easy to prove.

Proof of the main theorem: (5) $\Rightarrow$ (1)

- Proof by induction on n. Assume that the argument holds for $n < p \not< 4$.
- By the CCD property and being a cograph is preserved under complementation, we may examine G or $\bar{G}$.
  - WLOG, say $\bar{G}$ is disconnected.
- By the inductive hypothesis, each of the connected components of $\bar{G}$ is a cograph, thus $\bar{G}$ is a cograph (establishment by union).

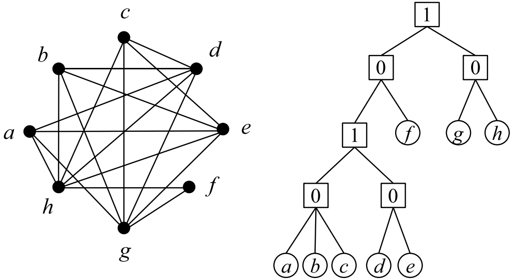
Outline

Graph isomorphism

- Graph isomorphism problem (GI): Given two graphs G and H, determine whether G and H are isomorphic.
  - It's easy to see that GI is in **NP**.
  - GI is in **P**? GI is in **NP-c**?
- If G, H are cographs, GI can be solved in linear time (by making use of standard cotrees) [Corneil et al. SIAM J. Comput. 1985].

★ Note that the *Induced subgraph* isomorphism problem is in **NP-c**, even for cographs.

Generating formula for cliques (kernels)



- For cliques: $(((a \vee b \vee c) \wedge (d \vee e)) \vee f) \wedge (g \vee h)$.
  - $(((1+1+1) \times (1+1)) + 1) \times (1+1) = 14$.
  - Cliques: *adg*, *adh*, *aeg*, *aeh*, *bdg*, *bdh*, *beg*, *beh*, *cdg*, *cdh*, *ceg*, *ceh*, *fg*, *fh*.
  - Size of the largest clique: $\max\{(\max\{1,1,1\} + \max\{1,1\}), 1\} + \max\{1,1\} = 3$.
- For kernels: $(((a \wedge b \wedge c) \vee (d \wedge e)) \wedge f) \vee (g \wedge h)$.
  - Kernels: *abcf*, *def*, *gh*.

Thank you!