Computing the girth of a planar graph in $O(n \log n)$ time

Oren Weimann and Raphael Yuster

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Outline

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Girth

Definition (The girth of a graph G)

The length of the shortest cycle of G.

The girth has tight connections to many graph properties.

- chromatic number;
- minimum or average vertex-degree;
- diameter;
- connectivity;
- genus;
- ...

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The road of computing the girth of a graph

For general graphs $G = (V, E)$, $n = |V|$ and $m = |E|$:

- $O(nm)$ [Itai & Rodeh, *SIAM J. Comput.* 1978].
  - $O(n^2)$ with an additive error of one.

For computing the shortest **even-length** cycle:

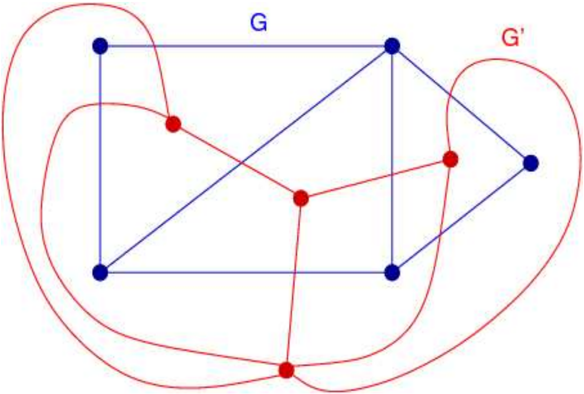
- $O(n^2\alpha(n))$ [Monien, *Computing* 1983].
- $O(n^2)$ [Yuster & Zwick, *SIAM J. Discrete Math.* 1997].

The road of computing the girth of a graph (contd.)

For planar graphs:

- $O(n)$ if the girth is bounded by **3** [Papadimitriou & Yannakakis, *Inform. Process. Lett.* 1981].
- $O(n)$ if the girth is bounded by a **constant** [Eppstein, *J. Graph Algorithms Appl.* 1999].
- $O(n^{5/4} \log n)$ [Djidjev, *ICALP'2000*]
- $O(n \log^2 n)$ [implicitly by Chalermsook et al., *SODA'2004*]
- $O(n \log n)$ [Weimann & Yuster, *SIAM J. Discrete Math.*, 2010]

A planar graph & its dual plane graph

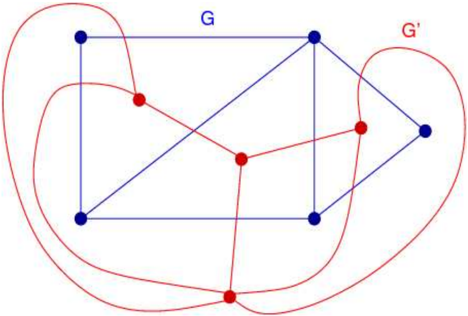


- a cut in G (resp., G') $\Leftrightarrow$ a cycle in G' (resp., G)

Outline

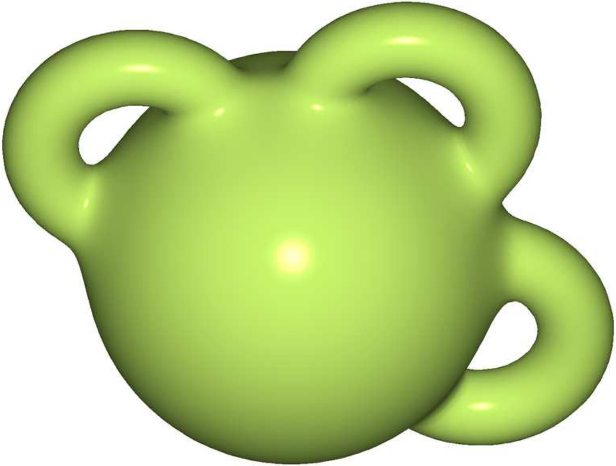
Planar graphs

- Planar embedding.



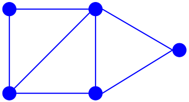
- point? curve? face?
- genus?

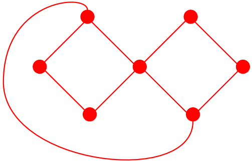
Genus = minimum number of handles

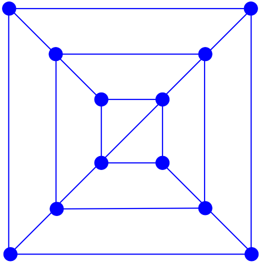


(k-)outerplanar graphs

- outerplanar: all the vertices lie on a single face.
- k-outerplanar: deletion of the vertices on the outer face results in a $(k-1)$-outerplanar graph.







Some important bounds on planar graphs

Euler's formula

A graph embedded on an orientable surface of genus g with n vertices, m edges, and f faces satisfies

$$n - m + f \geq 2 - 2g.$$

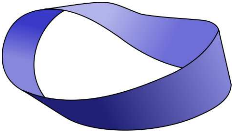


Fig.: An example of a non-orientable surface.

Theorem

A connected planar graph with $n \geq 3$ vertices, m edges and f faces satisfies $m \leq 3n - 6$ and $n - m + f = 2$.

Definition (Separator)

A separator is a set of vertices whose removal leaves connected components of size $\leq 2n/3$.

Theorem

- *If* G *is a planar graph, then it has a separator of* $O(\sqrt{n})$ *vertices.*
- *If* G *has genus* $g > 0$*, then it has a separator of* $O(\sqrt{gn})$ *vertices that can be found in* $O(n + g)$ *time.*
- *Every* k*-outerplanar graph has a separator of size* $O(k)$ *that can be found in* $O(n)$ *time.*

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An upper bound on the girth of a graph

- Given an embedded planar graph G, the size of each face is clearly an upper bound on G's girth.
- However, the shortest cycle is NOT necessarily a face.
- Nevertheless, the minimum face size helps us to compute the girth.

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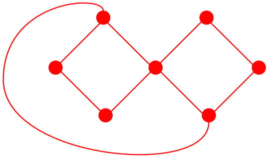

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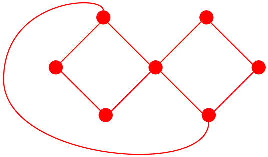

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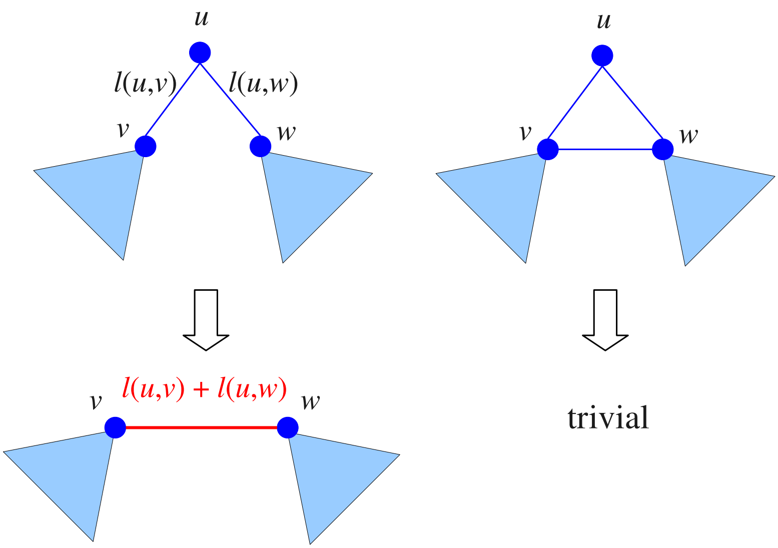
Outline

Stage 1: $G \Rightarrow G'$

- Some assumptions on G:
  - G is 2-connected ($\Rightarrow$ no vertex has degree 0 or 1).
    - Otherwise we can run the algorithm on each 2-connected component separately.
  - G is not a simple cycle (trivial case).
- Modify G to G' such that each edge is incident with a vertex of degree ≥ 3.

Stage 1: $G \Rightarrow G'$ (contd.)



Stage 1: $G \Rightarrow G'$ (contd.)

- $\text{girth}(G)$ = the length of the shortest cycle of G'.
- h: the minimum face-size of any embedding of G.
  - the number of edges on a shortest cycle of G' is also bounded by h.
    - $\because$ $\text{girth}(G) \leq h$ and only edge contractions from G to G' are performed.
- G' has nonnegative edge-lengths.

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Lemma 2.1

G' has at most $36n/h$ vertices.

▸ The proof

- The lemma provides a way to compute an upper bound h for the minimum face-size of any embedding of G.
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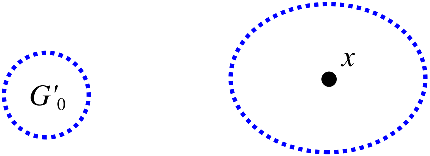
Stage 2: Cover G' by k-outerplanar graphs



- x: an arbitrary vertex in G'; let $k = 2h$.
- G'_0: the graph induced by the vertices with distance from x between 0 and k.

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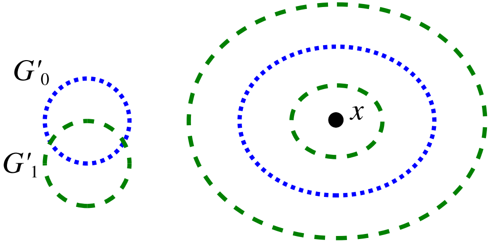

- x: an arbitrary vertex in G'; let $k = 2h$.
- G'_1: the graph induced by the vertices with distance from x between $k/2$ and $3k/2$.

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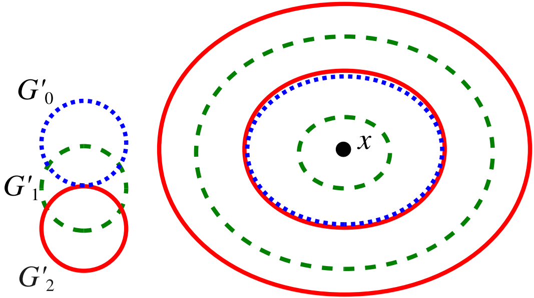

- x: an arbitrary vertex in G'; let $k = 2h$.
- G'_2: the graph induced by the vertices with distance from x between k and $2k$.

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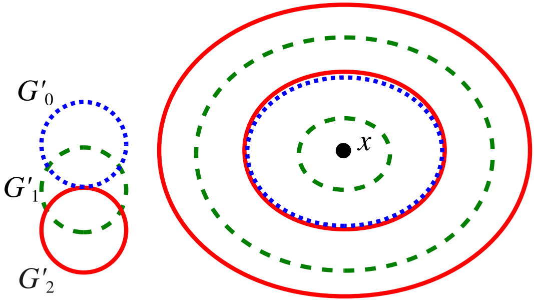

- x: an arbitrary vertex in G'; let $k = 2h$.
- G'_i: the graph induced by the vertices with distance from x between $i \cdot k/2$ and $k + i \cdot k/2$ for $i = 0, 1, \ldots, \frac{2(n-k)}{k}$.

Some facts about G_i''s:

- Every G_i' is a $(k+1)$-outerplanar graph.
- Every G_i' overlaps with at most two other graphs, G_{i-1}' and G_{i+1}'.
- The shortest cycle must be entirely contained within a single G_i'.

Stage 3: Run the k-outerplanar graph algorithm on G_i''s

- Run the algorithm for k-outerplanar graphs on every G_i' separately to find its shortest cycle and return the shortest one among them.
    - Each run requires $O(k|G_i'|\log|G_i'|)$ time (a divide-and-conquer algorithm).
- The total time complexity is thus

$$\sum_i c \cdot k|G_i'|\log|G_i'| \leq c \cdot 2h\log n \cdot \sum_i |G_i'| = O(n\log n).$$

- Notice that every vertex in G_i' appears in at most three G_i''s $\Rightarrow \sum_i |G_i'| = O(|G'|) = O(n/h)$.

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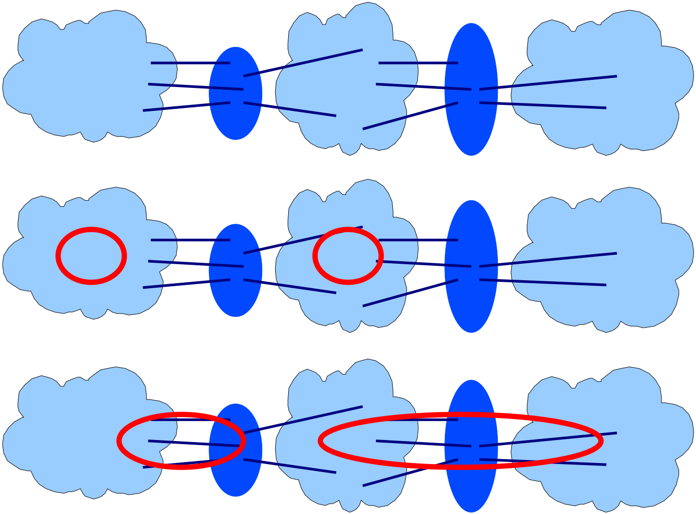
Where are the shortest cycles?

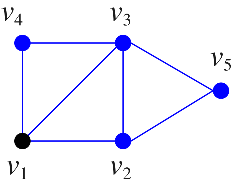


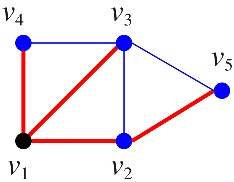
Theorem (Henzinger et al., *J. Comput. Sys. Sci.* 1997)

There is an $O(n)$ algorithm for a planar graph G with nonnegative edge-lengths to compute the distances from a given source v to all vertices of G.

- It takes $O(kn)$ time to construct the *shortest-path tree* from every separator vertex of a k-outerplanar graph.

A shortest-path tree from v_1





Computing the shortest cycle passing a designated vertex

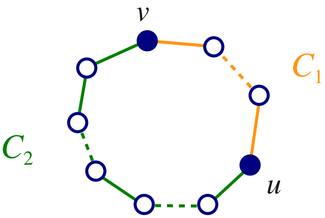
Lemma 3.1

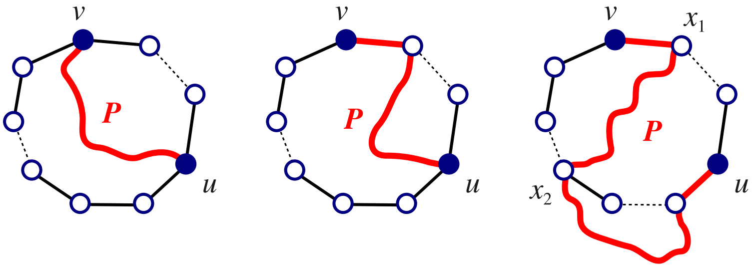
Let G be a connected graph with nonnegative edge-lengths. If

- *a vertex v lies on a shortest cycle, and*
- *T is a shortest-path tree from v,*

then there is a shortest cycle that passes through v and has **exactly one** *edge not in T.*

- C: the shortest cycle passing through v with the fewest number (say $\ell \geq 2$) of edges not in T.





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- a vertex v lies on a shortest cycle, and
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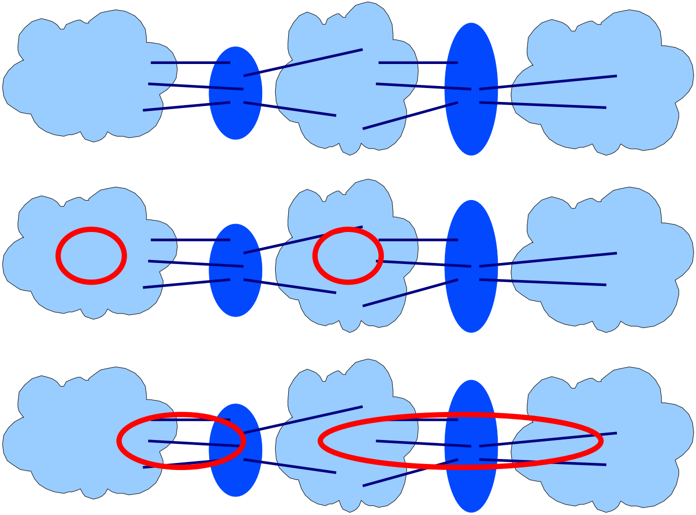
then there is a shortest cycle that passes through v and has **exactly one** edge not in T.

- It suggests an $O(n)$-time procedure to find the shortest cycle passing a given vertex v.
  - For each edge (x, y) not in T whose length is $\ell(x, y)$, we look at $\text{dist}_v(x) + \text{dist}_v(y) + \ell(x, y)$.
  - Take the minimum of this sum over all edges (x, y) not in T.



The $O(kn \log n)$ algorithm for k-outerplanar graphs

Assume that the removal of the separator results in $t \geq 2$ connected components.

$$T(n) = T(n_1) + T(n_2) + \ldots T(n_t) + O(kn),$$

where $\sum_{i=1}^{t} n_i \leq n$ and every $n_i \leq 2n/3$.

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Thank you.

Proof of Lemma 2.1

Fix an embedding of G with minimum face size h. Say:

G has n vertices, m edges, and f faces, and
G' has n' vertices, m' edges, and f' faces.

F: denote the set of faces in G;
$|x|$: the size of a face $x \in F$.

- It is easy to see that $f = f'$.
- $2m = \sum_{x \in F} |x| \geq \sum_{x \in F} h = fh$.
  - ▷ $f' = f \leq 2m/h \leq 6n/h$ ($\because m \leq 3n - 6$ for planar G).

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Let $S := \{v \in V(G) \mid \deg_G(v) \geq 3\}$ and $s = |S|$.

- ⋆ $m' \leq \sum_{v \in S} \deg_G(v)$.
- ⋆ $2(n' - s) + \sum_{v \in S} \deg_G(v) = 2m'$.
  - ▷ $\sum_{v \in S} \deg_G(v) = 2(m' - n' + s)$.

- By Euler's formula, we have $m' = n' + f - 2 \leq n' + 6n/h$.
  - ▷ $m' - n' \leq 6n/h$.

- Thus, $(\deg_G(v) \geq 3 \text{ for } v \in S)$

$$m' \leq \sum_{v \in S} \deg_G(v) \leq 3 \sum_{v \in S} (\deg_G(v) - 2) = 6(m' - n') \leq 36n/h.$$

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- ⋆ $2(n' - s) + \sum_{v \in S} \deg_G(v) = 2m'$.
  - ▷ $\sum_{v \in S} \deg_G(v) = 2(m' - n' + s)$.

- By Euler's formula, we have $m' = n' + f - 2 \leq n' + 6n/h$.
  - ▷ $m' - n' \leq 6n/h$.

- Thus, ($\deg_G(v) \geq 3$ for $v \in S$)

$$m' \leq \sum_{v \in S} \deg_G(v) \leq 3 \sum_{v \in S} (\deg_G(v) - 2) = 6(m' - n') \leq 36n/h.$$

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Proof of Lemma 2.1 (contd.)

Let $S := \{v \in V(G) \mid \deg_G(v) \geq 3\}$ and $s = |S|$.

- ⋆ $m' \leq \sum_{v \in S} \deg_G(v)$.
- ⋆ $2(n' - s) + \sum_{v \in S} \deg_G(v) = 2m'$.
  - ▷ $\sum_{v \in S} \deg_G(v) = 2(m' - n' + s)$.

- By Euler's formula, we have $m' = n' + f - 2 \leq n' + 6n/h$.
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$$m' \leq \sum_{v \in S} \deg_G(v) \leq 3 \sum_{v \in S} (\deg_G(v) - 2) = 6(m' - n') \leq 36n/h.$$

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